Published by Institute of Physics Publishing for SISSA

Received: *July 25, 2007*
Accepted: *August 28, 2007*
Published: *September 11, 2007*

Holomorphic anomaly in gauge theories and matrix models

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Abstract: We use the holomorphic anomaly equation to solve the gravitational corrections to Seiberg-Witten theory and a two-cut matrix model, which is related by the Dijkgraaf-Vafa conjecture to the topological B-model on a local Calabi-Yau manifold. In both cases we construct propagators that give a recursive solution in the genus modulo a holomorphic ambiguity. In the case of Seiberg-Witten theory the gravitational corrections can be calculated to all genus as quasimodular functions of $\Gamma(2)$. In the matrix model we fix the holomorphic ambiguity up to genus two. The latter result establishes the Dijkgraaf-Vafa conjecture at that genus and yields a new method for solving the matrix model at fixed genus in closed form in terms of generalized hypergeometric functions.

Keywords: Topological Field Theories, Supersymmetric gauge theory, Matrix Models, 1/N Expansion.

JHEP09(2007)054

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1. Introduction

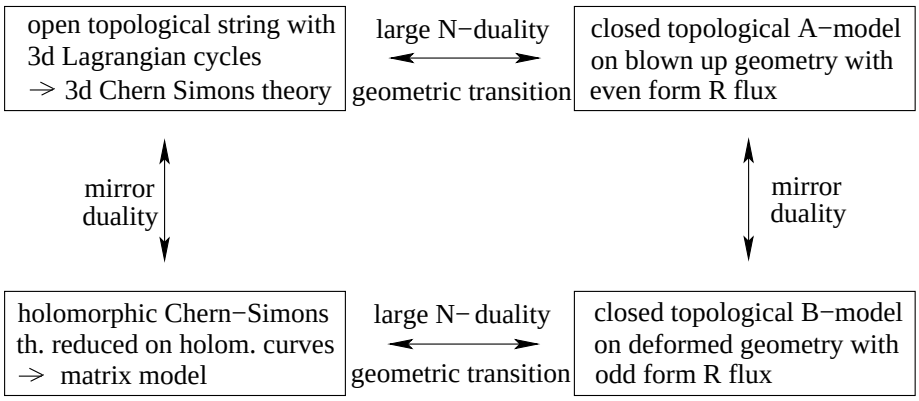
The holomorphic anomaly equations, discovered in [1] in a world-sheet analysis of a topological twisted σ-model known as B-model, are a generalization of Quillens anomaly to higher genus and more general world-sheet theories. Gauge theory is embedded in string theory and in the context of $N = 2$ geometric engineering the former can be obtained in a double scaling decoupling limit of the string theory. The holomorphic anomaly equations commute with the decoupling limit and the recursive procedure which determines the higher $F^{(g)}(\tau, \bar{\tau})$ [1] can be build up entirely from gauge theory quantities. The $F^{(g)}(\tau, \bar{\tau})$ describe certain gravitational couplings to the gauge theory.

In section 2 the corresponding topological partition of the 4d $N = 2$ SUSY gauge is determined by integrating the holomorphic anomaly recursively, up to finitely many data, which have to be fixed by analyzing the boundary behavior of the $F^{(g)}(\tau, \bar{\tau})$. The properties of $F^{(g)}(\tau, \bar{\tau})$ are to a large extend determined by the modular group of the corresponding Seiberg-Witten curve and we are able to write them as global expressions in terms of

"almost holomorphic" modular functions of this group. Various holomorphic limits are readily taken from our expressions and provide conjectural solutions to the unitary matrix model [3].

Riemann surfaces Σ_g of all genera can be embedded in noncompact Calabi-Yau threefolds X, i.e. the local limit of the B-model exist for an arbitrary Riemann surface. There are choices in this embedding, which affect the reduction of the holomorphic $(3,0)$ form from X to a one form λ on Σ_g, which is a key datum of the local B-model limit. Except for the product case, λ will be a meromorphic form with residua on the punctures of a Riemann surface $\Sigma_{g,h}$. In the pure SW-case, one has poles with vanishing residua. In the massive SW-case and other local geometries, one has further mass parameters associated with the non-vanishing residua.

In section 3 we apply the holomorphic anomaly equations to $N = 1$ 4d gauge theory in a geometry proposed by [4]. This is a case where λ has essential singularities or differently put one deals with open Riemann surfaces. According to the Dijkgraaf-Vafa correspondence [5] the holomorphic anomaly equations provide here in a particular region solutions to the large N expansion of a complex matrix model, which is the solution to an open string problem. This has a description in the A and the B language as summarized following commuting diagram of duality relations



The Dijkgraaf-Vafa conjecture, relevant to calculate effective terms in $\mathcal{N} = 1$ supersymmetric theories in 4d, involves non-compact Calabi-Yau geometries, which are in particular not toric. Explicit test of the conjecture are therefore difficult and have been done only at tree level [5] and at genus one [8, 10]. In the genus one case it is found that the free energy for multi-cut matrix models can be written in closed form [8, 11]. We set the B-model formalism up to get recursively closed expressions at all genus and check the Dijkgraaf-Vafa correspondence of the first non-trivial example, the two cut case, explicitly at genus two.

As in the $N = 2$ case, we make a detailed analysis of the moduli space and the modular transformation of the periods in appendix A, which enables us to solve the theory at various regions in the moduli space. Different from in the $N = 2$ case however there is a divisor with essential singularities of the periods in the moduli space, where the perturbative

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description breaks down. From the topological string point of view we obtain solutions for a new class of non-toric, non-compact Calabi-Yau geometries, which have no point of maximal unipotent monodromy.

2. Gravitational corrections to N=2 Seiberg-Witten Gauge theory

In this section we introduce a B-model, which calculates the higher genus space-time instantons of $N = 2$ Seiberg-Witten gauge theory. The genus g generating function $\mathcal{F}^{(g)}(a)$ of these space-time instantons describe gravitational couplings to the gauge theory [14]. The coefficients of

$$\mathcal{F} = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(a) \tag{2.1}$$

can be calculated iteratively in the instanton number, $1/a^4$ powers below, using localization in the moduli space of gauge theory [15, 16, 18, 17] or worldsheet instantons [8, 9]. Global properties under the monodromy group, which should be central in the solution of the theory, remain obscure in these approaches. We find that studying the global properties enables us to solve the B-model completely and in particular fix the holomorphic ambiguity. This yields closed modular expressions which determine the instanton contributions to all orders in the instanton number, but are iteratively in the genus i.e. in λ. We finish the section with some speculations how to obtain closed expressions in λ as well.

2.1 Modular properties of the genus zero and genus one sector

We focus on simplest case of SU(2) gauge theory without matter. Generalizations to other gauge groups and matter spectra with asymptotic freedom are certainly possible. The monodromy group of pure SU(2) Seiberg-Witten theory is $\Gamma(2) \in \Gamma_0 = \mathrm{SL}(2,\mathbb{Z})$ generated by [19]

$$M_\infty = PT^{-2}, \qquad M_1 = ST^2S\,, \tag{2.2}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the generators of Γ_0 and $P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Using modular properties as well as the B-model holomorphic anomaly we are able to express the $\mathcal{F}^{(g)}$ in terms of $\Gamma(2)$ forms and the quasi-modular form of weight two E_2, see (2.22) for our conventions.

The natural embedding of gauge theory in type IIB string theory [20] is the physical explanation that the higher genus worldsheet technique of [1] applies to the gauge theory space-time instanton calculation. More precisely as found in [20] IIB theory on the local Calabi-Yau space $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ has a double scaling limit in the two complexified Kähler parameters t_1, t_2 of $\mathbb{P}^1 \times \mathbb{P}^1$ in which $\mathcal{F}^{(g)}(t_1,t_2)$ approaches $\mathcal{F}^{(g)}(a)$. What we find below is that the holomorphic anomaly equation make sense in the limit and can be directly viewed as a property of the gauge theory.

The Picard-Fuchs equation for the periods $a = \int_a \lambda$ and $a_D = \int_b \lambda$ of the elliptic curve [19]

$$y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2) \tag{2.3}$$

with meromorphic differential $\lambda = \frac{\sqrt{2}}{2\pi}\frac{y}{x^2-1}\mathrm{d}x$

$$\frac{\partial^2 a}{\partial^2 u} = \frac{a}{4(1-u^2)} \tag{2.4}$$

allows to calculate the genus zero prepotential using the relation $\frac{\partial \mathcal{F}^{(0)}}{\partial a} = a_D$ up an irrelevant constant. Below we set the Seiberg-Witten scale $\Lambda = 1$ for convenience. The scale dependence can be easily recovered by dimensional analysis. The details of the calculation that leads to explicit expressions for the periods can be found[1] in [21]. The elliptic curve (2.3) with $\Gamma(2)$ monodromy has the j-function $j(\tau) = \frac{(3+u^2)^3}{27(u^2-1)^2}$, which allows to write [22]

$$u(\tau) = \frac{c+d}{b}\,, \tag{2.5}$$

as the Hauptmodul of $\Gamma(2)$ in terms of the ratio $\tau = \int_b \frac{\mathrm{d}x}{y} / \int_a \frac{\mathrm{d}x}{y} = -\frac{1}{4\pi i}\frac{\partial^2 \mathcal{F}^{(0)}}{\partial^2 a}$ of the periods over the holomorphic one-form. The latter are solutions of the ${}_2F_1$ hypergeometric differential equation $\frac{\partial^2 a}{\partial^2 u} = \frac{2u}{(1-u^2)}\frac{\partial a}{\partial u} + \frac{a}{4(1-u^2)}$, which implies that the inverse relation to (2.5) can be written in terms of a Schwarz-triangle function (see e.g. [23])

$$\tau(u^2) = s(\frac{1}{4},\frac{1}{4},1;u^2) := \frac{i(1-u^2)^{\frac{1}{4}}}{2(1-i)}\frac{{}_2F_1(\frac{1}{4},\frac{1}{4},1;1-u^2)}{{}_2F_1(\frac{1}{4},\frac{1}{4},1;\frac{1}{1-u^2})}\,. \tag{2.6}$$

We defined in (2.5) $b = \theta_2^4$, $c = \theta_3^4$, $d = \theta_4^4$, where θ are the Jacobi θ-functions and the Jacobi relation reads $b + d = c$. Using the modular transformation

$$T: \begin{array}{l} b \to -b \\ c \leftrightarrow d \end{array} \qquad S: \begin{array}{r} b \to -\tau^2 d \\ d \to -\tau^2 b \\ c \to -\tau^2 c \end{array} \tag{2.7}$$

and (2.2) it follows immediately that $u(\tau)$ is invariant under $\Gamma(2)$. For further reference let us note that the discriminant of the elliptic curve is given by $u = \infty$ and

$$\Delta = u^2 - 1 = 4\frac{cd}{b^2} = 0\,. \tag{2.8}$$

The Kähler potential in rigid special geometry is given by (see [24] for a recent review)

$$K = i(X^I \bar{F}_{\bar{I}} - \bar{X}^{\bar{I}} F_I)\,. \tag{2.9}$$

For the Seiberg-Witten curve $X^1 = a$ and $\tau := \frac{\partial^2}{\partial^2 a}F$ so that the metric becomes

$$g_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} K = 2\tau_2\,. \tag{2.10}$$

The genus one amplitude is obtained by integrating the genus holomorphic anomaly using the special geometry relation [25]

$$F^{(1)} = -\log(\sqrt{\tau_2}\eta\bar{\eta})\,. \tag{2.11}$$

[1] In [21] the isogenous curve $y^2 = (x^2 - \tilde{u})^2 - \Lambda^4$ with the meromorphic differential $\lambda = \frac{i\sqrt{2}}{4\pi} 2x^2 \frac{\mathrm{d}x}{y}$ is used. This means that $a^{\mathrm{SW}} = a^{\mathrm{KLT}}$, $a_D^{\mathrm{SW}} = a_D^{\mathrm{KLT}}/2$, $\tau_D^{\mathrm{SW}} = \tau_D^{\mathrm{KLT}}/2$ and the monodromy is $\Gamma_0(4)$ instead of $\Gamma(2)$.

Note that the holomorphic ambiguity is fixed by requiring that $F^{(1)}$ is invariant under $\mathrm{SL}(2,\mathbb{Z})$ transformations and regular inside the fundamental domain $\mathcal{H}$. Indeed η is the unique modular form of weight $\frac{1}{2}$ so that $F^{(1)}$ is regular inside $\mathcal{H}$.

In the holomorphic limit[2] $\bar{\tau} \to \infty$, we get [25]

$$\mathcal{F}^{(1)} = -\log(\eta(\tau)) = -\frac{1}{2}\log\left(\frac{\partial a}{\partial u}\right) - \frac{1}{12}\log(u^2-1)\ . \tag{2.12}$$

This the first equality agrees with the result in [14]. The form of $\mathcal{F}^{(1)}$ in terms of u and $\frac{\partial a}{\partial u}$ follows in the rigid limit from [25] and was observed in [8]. Using (2.12) and $\eta^{12} = 2^{-4}bcd$ one gets

$$\frac{\partial a}{\partial u} = \frac{\theta_2^2}{4}\ . \tag{2.13}$$

Further with (2.5) and the formulas for the derivatives of the θ functions we find

$$\mathcal{F}^{(0)}_{,3}(\tau) := \frac{\partial^3 \mathcal{F}^{(0)}}{\partial^3 a} = \frac{\partial \tau}{\partial a} = \frac{8\sqrt{b}}{cd} = \frac{1}{\Delta}\left(\frac{\partial u}{\partial a}\right)^3\ . \tag{2.14}$$

This can alternatively derived by taking twice the derivative w.r.t. to a in the Matone relation $\mathcal{F}^{(0)} - \frac{1}{2}a\frac{\partial \mathcal{F}^{(0)}}{\partial a} + u = 0$ [29]. Using the Picard-Fuchs (2.4) equation and (2.13) we can write the period as[3]

$$a(\tau) = \frac{1}{3\theta_2^2}(E_2 + c + d)\ . \tag{2.15}$$

Closely related expressions for the SW-periods of elliptic curves appear in [14] and for more general gauge groups in [27].

The weak-strong coupling duality element $S : \tau \to -\frac{1}{\tau}$ of $N = 4$ Super-Yang-Mills theory is not trivially realized in pure $N = 2$ Seiberg-Witten theory. It relates the gauge instanton expansion in the region of asymptotic freedom $\frac{1}{u}, \frac{1}{a} \to 0$ to the magnetic $\mathrm{U}(1)$ which is weakly coupled at the magnetic monopole point $z = (u-1)/2, a_D \to 0$. The gauge coupling of the asymptotic free theory is determined by $\tau(a)$, while the one of the magnetic $\mathrm{U}(1)$ is determined by $\tau_D(a_D)$ with the relation

$$\tau_D = -\frac{1}{\tau} \tag{2.16}$$

We note that the one-loop amplitude (2.11) is S-duality invariant. The holomorphic limit at the monopole points is taken by $\bar{\tau}_D \to \infty$. Therefore one has

$$\begin{aligned}\mathcal{F}^{D(1)}(a_D) =& -\log(\eta(\tau_D)) = -\frac{1}{2}\log\left(\frac{\partial a_D}{\partial u}\right) - \frac{1}{12}\log(u^2-1)\\ =& \frac{\log(2)}{3} - \frac{i}{6}\pi - \frac{\log(\tilde{a}_D)}{12} - \frac{\tilde{a}_D}{2^5} - \frac{3\,\tilde{a}_D^2}{2^9} - \frac{19\,\tilde{a}_D^3}{2^{12}3} + O(\tilde{a}_D)^4\end{aligned} \tag{2.17}$$

[2]Because of the volume of the diffeomorphism group generated by the Killing vector field on T^2 we need regularization of an infinite constant in that limit. Only the immediately well-defined $\partial_\tau F^{(1)}$ plays a rôle in the following. To define that limit consider τ and $\bar{\tau}$ as independent variables.

[3]To keep the notation simpler we suppress normalization factors of $\frac{1}{2\pi i}$ from the periods, which make them an integral representation of (2.2).

We derive, similar as (2.15) was obtained, from the first line of (2.17) a formula for the second period

$$a_D(\tau_D) = -\frac{i}{3\theta_4^2}(E_2 - b - c) \ . \tag{2.18}$$

This can then be inverted to obtain the series expansion in the second line of (2.17), where we rescale

$$\tilde{a}_D := i\frac{a_D}{2} \ . \tag{2.19}$$

It is naturally to define an anholomorphic period

$$A(\tau, \bar{\tau}) = \frac{1}{3\theta_2^2}(\hat{E}_2(\tau, \bar{\tau}) + c + d), \tag{2.20}$$

with the anholomorphic weight one form

$$\hat{E}_2(\tau, \bar{\tau}) := E_2(\tau) + \frac{6i}{\pi(\bar{\tau} - \tau)} \ . \tag{2.21}$$

Different then the Eisenstein series E_4 and E_6, holomorphic modular forms of weight 4 and 6 which generate the ring of holomorphic forms of Γ_0, the holomorphic E_2 transforms not quite as modular form of weight 1, but rather quasi-modular under Γ_0, i.e. with a shift

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{12}{2\pi i}c(c\tau + d) \ . \tag{2.22}$$

However the anholomorphic piece in (2.20) cancels the shift so that $\hat{E}_2(\tau, \bar{\tau})$ transforms as a form of weight 2.

It follows from (2.20) that $a(\tau) = \lim_{\bar{\tau}\to\infty} A(\tau, \bar{\tau})$ and $a_D(\tau_D) = \lim_{\bar{\tau}_D\to\infty} \frac{1}{\tau_D} A(-\frac{1}{\tau_D}, -\frac{1}{\bar{\tau}_D})$. Moreover S-duality transformations (2.7) allow us to write a and a_D as functions of τ or τ_D. In particular we can integrate (2.15) w.r.t. a_D in the magnetic phase to obtain

$$\mathcal{F}^{D(0)} = \frac{1}{2}\tilde{a}_D^2 \log\left(-\frac{\tilde{a}_D}{2}\right) + 4\tilde{a}_D - \frac{3{\tilde{a}_D}^2}{2^2} + \frac{\tilde{a}_D^3}{2^4} + \frac{5\,\tilde{a}_D^4}{2^9} + \frac{11\,\tilde{a}_D^5}{2^{12}} + O(\tilde{a}_D^6) \ . \tag{2.23}$$

2.2 Propagators and integrating the holomorphic anomaly equations

From the genus zero and genus one amplitude one can derive the propagator [1]. With the simplification for local B-model explained in [26] we obtain

$$S^{aa} = S = \frac{2F^{(1)}_{,1}}{F^{(0)}_{,3}} = \frac{1}{24}\left(E_2(\tau) - \frac{3}{\pi\tau_2}\right) =: \frac{1}{24}\hat{E}_2(\tau, \bar{\tau}) \,, \tag{2.24}$$

which in the holomorphic limit becomes

$$\mathcal{S}^{aa} = \mathcal{S} = \frac{2\mathcal{F}^{(1)}_{,1}}{\mathcal{F}^{(0)}_{,3}} = \frac{1}{24}E_2(\tau) \ . \tag{2.25}$$

This quantity contains the contribution from the boundaries of the WS moduli space in the topological string theory. It is closely related to $T_{k,l}$ the quantity that arises if one considers

correlators of integrated two-form operators constructed from the descent equation in the gauge theory on four manifolds. More precisely intersections of the corresponding 2-cycles require contact terms which are $T_{k,l} = S^{a_i a_j} \frac{\partial u_k}{\partial a_i} \frac{\partial u_l}{\partial a_j}$, see [28].

Equations (2.14), (2.11), (2.24) define the data needed to recursively solve the B-model recursively [1]. Using the fact that the formalism of integrating the holomorphic anomaly equations commutes with the double scaling limit taken to obtain the gauge theory [20] and power counting in the propagator in the Feynmann rules of [1] we obtain the following general result

$$F^{(g)}(\tau,\bar\tau) = X^{g-1} \sum_{k=0}^{3(g-1)} \hat{E}_2^k(\tau,\bar\tau) c_k^{(g)}(\tau) \ . \tag{2.26}$$

Here we defined $X = \frac{b}{1728 c^2 d^2} = \frac{1}{108(u^2-1)^2 b^3}$, which transforms as a weight -6 object $X(\frac{a\tau+b}{c\tau+d}) \to \frac{1}{(c\tau+d)^6} X(\tau)$ under $\Gamma(2)$. $F^{(g)}$ is invariant under $\Gamma(2)$, which implies that $c_k^{(g)}$ are homogeneous of weight $6(g-1)-2k$ in (b,c,d). Further conditions on $c_k^{(g)}$ come from regularity of $F^{(g)}(t,\bar t)$ at $u=0$ and a gap condition at the conifold $u=1$ as dicussed below.

We obtain the holomorphic limits of the expansion in the asymptotic freedom and the strong coupling region as

$$\mathcal{F}^{(g)}(a) = \lim_{\bar\tau\to\infty} F^{(g)}(\tau,\bar\tau), \quad \text{and} \quad \mathcal{F}^{D(g)}(a_D) = \lim_{\bar\tau_D\to\infty} F^{(g)}\left(-\frac{1}{\tau_D}, -\frac{1}{\bar\tau_D}\right) \tag{2.27}$$

and the a or a_D ($\tilde a_D$) expansion are obtained by inverting (2.15) or (2.18). We note that the leading behavior of electric and magnetic expansion in these parameters is

$$\mathcal{F}^{(g)}(a) \sim \frac{(-1)^g B_{2g}}{g(2g-2)(2a)^{2g-2}} \quad \text{and} \quad \mathcal{F}^{D(g)}(a_D) \sim \frac{B_{2g}}{2g(2g-2)\tilde a_D^{2g-2}} \tag{2.28}$$

respectively. The first asymptotic behavior can be derived in the gauge theory limit of type II theory on $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$. More precisely one uses in the Gopakumar-Vafa expansion $\frac{1}{\left(2\sin\frac{m\lambda}{2}\right)^2} = \sum_{g=0} \lambda^{2g-2}(-1)^{(g-1)} \frac{B_{2g}}{2g(2g-2)!} m^{2g-2}$ and the multiplicity $n_{m,0}^{(g)} = \delta_{g,0}\delta_{m,0} = -2$ of BPS states corresponding to constant maps on one $\mathbb{P}^1$ as well as properties of the limit discussed in [20]. The derivation is similar as for the constant map contribution $\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}$ in [38]. The asymptotic in the magnetic expansion comes from the occurrence of the $c=1$ string at the conifold [40].

We come now to the explicite iterative solutions in the genus. For example the recursive definition of $F^{(2)}$ is

$$F^{(2)}(\tau,\bar\tau) = \frac{1}{2} S F_{,2}^{(1)} + \frac{1}{2} S (F_{,1}^{(1)})^2 + \frac{5}{24} S^3 (F_{,3}^{(0)})^2 - \frac{1}{8} S^2 F_{,4}^{(0)} - \frac{1}{2} S^2 F_{,1}^{(1)} F_{,3}^{(0)} + X c_0^{(2)}(u) \,, \tag{2.29}$$

where $c_0^{(2)}(u)$ is the holomorphic ambiguity at genus two. This ambiguity must be invariant under $\Gamma(2)$, which implies that it can be written in terms of u. Moreover regularity of $F^{(g)}$ at $u \to \infty$ and the leading pole behavior at $u \to 1$ implies that it is of the form

$$X^{g-1} c_0^{(g)}(u) = u^{3-g} \sum_{i=1}^{2g-2} \frac{A_i^{(g)}}{\Delta^i(u)} \ . \tag{2.30}$$

Note that $A_i^{(g)}$ are undetermined constants and the right hand is a rational function of the $\Gamma(2)$ invariant function u. Using (2.14), (2.11) and the standard formulas for the derivatives of θ_i, E_2 we may write (2.29) with $h = (b + 2d)$ as

$$F^{(2)}(\tau,\bar\tau) = \frac{X}{3}\left(5\hat E_2^3 - 9\hat E_2^2 h + 6\hat E_2(b^2+cd) - \frac{h(16b^2+19cd)}{10}\right), \tag{2.31}$$

an almost holomorphic modular function of $\Gamma(2)$. Here we determined the ambiguity as follows. Using (2.27), (2.15) we can expand (2.31) in the electric and magnetic holomorphic limits. With the leading behavior (2.28) we found $A_1^{(2)} = -\frac{19}{12960}$ and $A_2^{(2)} = -\frac{2}{405}$. This yields

$$\mathcal{F}^{(2)} = -\frac{1}{240\cdot 2\,a^2} - \frac{11}{2^{18}\,a^{10}} - \frac{117}{2^{22}\,a^{14}} - \frac{171201}{2^{34}\,a^{18}} + O\left(\frac{1}{a^{22}}\right) \tag{2.32}$$

This series predicts all genus 2 instantons and checks with the coefficients that appear in the literature [15, 8]. The expansion in a_D is obtained from (2.31) using (2.27), (2.7), (2.18)

$$\mathcal{F}^{D(2)} = -\frac{1}{240\,\tilde a_D^2} - \frac{\tilde a_D}{2^{13}} - \frac{13\,\tilde a_D^2}{2^{16}} - \frac{129\,\tilde a_D^3}{2^{17}5} + O(\tilde a_D^4) \tag{2.33}$$

Solving the recursion for genus 3 yields [26]

$$\begin{aligned}
F^{(3)} =\ & SF^{(2)}_{,1}F^{(1)}_{,1} - \frac{1}{2}S^2F^{(2)}_{,1}F^{(0)}_{,3} + \frac{1}{2}SF^{(2)}_{,2} + \frac{1}{6}S^3(F^{(1)}_{,1})^3F^{(0)}_{,3} - \frac{1}{2}S^2F^{(1)}_{,2}(F^{(1)}_{,1})^2 \\
& -\frac{1}{2}S^4(F^{(1)}_{,1})^2(F^{(0)}_{,3})^2 + \frac{1}{4}S^3(F^{(1)}_{,1})^2F^{(0)}_{,4} + S_2^3F^{(1)}_{,2}F^{(1)}_{,1}F^{(0)}_{,3} - \frac{1}{2}S^2F^{(1)}_{,3}F^{(1)}_{,1} \\
& -\frac{1}{4}S^2(F^{(1)}_{,2})^2 + \frac{5}{8}S^5F^{(1)}_{,1}(F^{(0)}_{,3})^3 - \frac{2}{3}S^4F^{(1)}_{,1}F^{(0)}_{,4}F^{(0)}_{,3} - \frac{5}{8}S^4F^{(1)}_{,2}(F^{(0)}_{,3})^2 \\
& +\frac{1}{4}S^3F^{(1)}_{,2}F^{(0)}_{,4} + \frac{5}{12}S^3F^{(1)}_{,3}F^{(0)}_{,3} + \frac{1}{8}S^3F^{(0)}_{,5}F^{(1)}_{,1} - \frac{1}{8}S^2F^{(1)}_{,4} - \frac{7}{48}S^4F^{(0)}_{,5}F^{(0)}_{,3} \\
& +\frac{25}{48}S^5F^{(0)}_{,4}(F^{(0)}_{,3})^2 - \frac{5}{16}S^6(F^{(0)}_{,3})^4 - \frac{1}{12}S^4(F^{(0)}_{,4})^2 + \frac{1}{48}S^3F^{(0)}_{,6} + X^2c_0^{(3)}(u)\,,
\end{aligned} \tag{2.34}$$

and we determined the coefficients $A_1^{(3)} = \frac{59}{2449440}$, $A_2^{(3)} = \frac{14669}{3265920}$, $A_3^{(3)} = \frac{4133}{204120}$ and $A_4^{(3)} = \frac{5359}{306180}$. This yields the following almost complex modular expression for $F^{(3)}$

$$\begin{aligned}
F^{(3)} =\ & 4X^2\Big(5\hat E_2^6 - 25\hat E_2^5 h + 40\hat E_2^4(2b^2+5cd) - \frac{1}{3}\hat E_2^3 h(529b^2+559cd) \\
& +\frac{\hat E_2^2}{5}(1172b^4+4060b^2cd+1223(cd)^2) - \frac{\hat E_2}{5}h(844b^4+1685b^2cd+310(cd)^2) \\
& +\frac{1}{210}(10718b^6+49596b^4cd+44007b^2(cd)^2+944(cd)^3)\Big)\,,
\end{aligned} \tag{2.35}$$

from which the electric

$$\mathcal{F}^{(3)} = \frac{1}{1008\cdot 2^3}\left(-\frac{1}{a^4} + \frac{441}{2^{12}a^{12}} + \frac{18459}{2^{16}a^{16}} + \frac{62106849}{2^{28}a^{20}} + \frac{256368735}{2^{31}a^{24}} + O\left(\frac{1}{a^{28}}\right)\right) \tag{2.36}$$

and magnetic expansions

$$\mathcal{F}^{D(3)} = \frac{1}{1008}\frac{1}{\tilde a_D^4} - \frac{9\,\tilde a_D}{2^{20}} - \frac{143\,\tilde a_D^2}{2^{22}} - \frac{63827\,\tilde a_D^3}{2^{27}7} + O(\tilde a_D^4) \tag{2.37}$$

follow from (2.27). The number of terms in the modular expressions of $F^{(g)}$ grow much slower then the number of graphs in the holomorphic anomaly equation, because many graph contributions are proportional to the same quasimodular form. For genus four, where the holomorphic anomaly equation has 83 graphs we find

$$\begin{aligned}F^{(4)} = 4X^3\Big\{ &\frac{1150}{9}\hat{E}_2^9 - 985\hat{E}_2^8 h + 2\hat{E}_2^7(2399b^2+7001cd) \\ &-\frac{14}{3}\hat{E}_2^6 h(3761b^2+6125cd) + \frac{3}{5}\hat{E}_2^5(85863b^4+363344b^2cd+240083(cd)^2) \\ &-\frac{1}{5}\hat{E}_2^4 h(604469b^4+1677340b^2cd+547811(cd)^2) \\ &+\frac{2}{5}\hat{E}_2^3(531266b^6+2793615b^4cd+3285123b^2c^2d^2+447656(cd)^3)) \\ &-\frac{2}{35}\hat{E}_2^2 h(4430756b^6+16550337b^4cd+11925927b^2(cd)^2+889964(cd)^3) \\ &+\frac{1}{175}\hat{E}_2(31232428b^8+195274840b^6cd+329613819b^4(cd)^2 \\ &+130729960b^2(cd)^3+3566728(cd)^4) - \frac{1}{1575}h(87826748b^8+423770948b^6cd \\ &+511313601b^4(cd)^2+128098172b^2(cd)^3+4442006(cd)^4)\Big\}\end{aligned} \tag{2.38}$$

with electric

$$\mathcal{F}^{(4)} = -\frac{1}{1440\cdot 2^5}\left(\frac{1}{a^6}+\frac{765}{2^{12}a^{14}}+\frac{126195}{2^{16}a^{18}}+\frac{1925006715}{2^{29}a^{22}}+\frac{14420664765}{2^{32}a^{26}}+\mathcal{O}\left(\frac{1}{a^{30}}\right)\right) \tag{2.39}$$

and magnetic expansion

$$\mathcal{F}^{D(4)} = -\frac{1}{1440\tilde{a}_D^6}+\frac{1125\,\tilde{a}_D}{2^{29}}-\frac{3915\,\tilde{a}_D^2}{2^{28}}+\frac{4786021\,\tilde{a}_D^3}{2^{35}}+O(\tilde{a}_D^4) \tag{2.40}$$

We derived expressions for $F^{(g)}$ in terms of modular forms up to genus six and checked the large a expansions against results made available to us by Nakajima.[4] Let us report here the dual expansions, which are interesting as they correspond to perturbations of the $c=1$ string at the selfdual radius by momentum operators

$$\mathcal{F}^{D(5)} = \frac{1}{1056\,\tilde{a}_D^8}-\frac{77175\,\tilde{a}_D}{2^{36}}-\frac{100971\,\tilde{a}_D^2}{2^{32}}-\frac{5142558213\,\tilde{a}_D^3}{2^{43}11}+O(\tilde{a}_D^4)\,, \tag{2.41}$$

$$\mathcal{F}^{D(6)} = -\frac{691}{327600\,\tilde{a}_D^{10}}-\frac{18753525\tilde{a}_D}{2^{44}}-\frac{16908525\tilde{a}_D^2}{2^{40}}-\frac{672990085791\tilde{a}_D^3}{2^{49}13}+O(\tilde{a}_D^4)\,. \tag{2.42}$$

2.3 Fixing the ambiguity

In (2.26) all $c_k^{(g)}$ except for $c_0^{(g)}$, the holomorphic ambiguity, are determined by the recursion relations (2.29), (2.34) that follow from the holomorphic anomaly equation in terms of lower genus $F^{(g)}$. Generally fixing the holomorphic ambiguity is a major problem in the

[4]These somewhat lengthy expressions are available on request.

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B-model, which however the case at hand is completely solvable. The discriminant locus of the curve (2.3) is at $u = \infty$ and at $u = \pm 1$ where (2.3) develops a node. At all other points in the moduli space of the Seiberg-Witten geometry $F^{(g)}$ must be regular. As follows from the global properties of the θ functions and E_2, regularity at $u = 0$ restricts the form of $c_k^{(g)}(\tau)$ to

$$c_k^{(g)} = h^{\frac{(1+(-1)^{g+k})}{2}} P_{d(g,k)}(b^2, cd), \tag{2.43}$$

where $P_{d(g,k)}(b^2, cd)$ is an homogeneous polynomial in b, c, d of degree

$$d(g,k) = 3(g-1) - k - \frac{(1+(-1)^{g+k})}{2}, \tag{2.44}$$

with $\frac{d(g,k)}{2} + 1$ coefficients. In particular in the ambiguity $c_0^{(g)}$ the number of unknown coefficients of this polynomial grows with $\sim \frac{3}{2}g$ slower then the number of the $A_i^{(g)}$ in (2.30), which grows with $\sim 2g$.

Moreover the leading terms in (2.33), (2.37), (2.40), (2.41), (2.42) correspond to correlators of the cosmological constant operator of the $c = 1$ matrix model in the genus g vacuum sector [40]. Very important is the occurrence of the *gap* in $\mathcal{F}^{(g)D}$, i.e. the absence of terms $\frac{1}{\tilde{a}_D^k}$, $k = 0, \ldots, 2g-3$. The $2g-2$ gap conditions fix the $A_i^{(g)}$, $i = 1, \ldots, 2g-2$ constants and hence the ambiguity (2.30). Together with the anomaly equation this provides a very efficient way to solve the theory completely. The asymptotic (2.28) and the particular form of (2.43) are further consistency constraints confirming the gap property. A gap follows if there is a matrix model $\frac{1}{N}$ expansion for the holomorphic topological string at a critical point, as e.g. at the orbifold point in the local $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ model discussed [7]. In this case the measure integration yields at each genus the negative power term in expansion parameter and the perturbative terms start with positive powers. This particular behavior of the magnetic expansion of the N=2 pure SU(2) model at the conifold is hence explainable by the proposed unitary matrix model [3]. As we saw above the model is solved by the gap property and the holomorphic anomaly. If one would like to employ $\frac{1}{N}$ techniques in order to determine the weak coupling instanton expansion one would have to do exactly what we have done in (2.26) namely to write the result globally. More generally the gap can be understood from the absence of correlators of the ground ring operators [30] in the $c = 1$ string describing the limit of the topological string near the nodal singularity (conifolds) of local models. Indeed we have checked that the gap occurs also at the conifold in the local $\mathcal{O}(-3) \to \mathbb{P}^2$ geometry and fixes the ambiguity of this model. How this extends generally to singularities of local models and the modifications for singularities of global models will be discussed in [51].

If we absorb X into genus expansion parameter λ in (2.26), then (2.1) becomes a sum of a quasi-modular forms. The simplest example of such an expansion, where the coefficients are however quasi modular forms of the full modular group Γ_0 appears in Hurwitz theory on T^2 [32]. As reviewed there this leads directly to combinatorial problem that is solved by free fermions and $Z = e^F$ can be written in a product form that has been recognized as a generalized θ-function product form in [33]. More examples of such product forms are provided by vertex algebras [34, 35] and arise in heterotic type II duality [36 – 39]. It

would be interesting to see whether the SW partition function is related to a generalized $\Gamma(2)$ theta function.

3. Dijkgraaf-Vafa conjecture

In [5], Dijkgraaf and Vafa proposed a remarkable relation between B-model topological string on a non-compact Calabi-Yau geometry and a matrix model.

3.1 Dijkgraaf-Vafa transition and geometric engineering

The n cut matrix model is obtained by reducing holomorphic Chern-Simons theory on D5-branes wrapping n $\mathbb{P}^1$'s in a modification of the geometry $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{P}^1$. The k-th $\mathbb{P}^1$ is wrapped by N_k branes, $k = 1, \ldots, n$. In the modified geometry the location of the $\mathbb{P}^1$'s in the originally flat $\mathcal{O}(0)$ x-direction is now fixed at the minima of a potential $W(x)$ of degree $n+1$. e.g. the $n = 1$ geometry is the blown up conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. The reduction yields a complex bosonic matrix model with the matrix potential[5] $W'(x)$, that needs as additional data the choice of contour for the eigenvalue integration [5].

The B-model geometry emerges after a transitions in which the n $\mathbb{P}^1$'s are blown down and deformed to S^3's. It has a local description as a hypersurface in $\mathbb{C}^4$

$$vw = W'(x)^2 + f(x) + y^2 \tag{3.1}$$

where x, y, v, w are coordinates of $\mathbb{C}^4$, $f(x)$ is polynomial of degree $n-1$ that splits the n double zeros of $W'(x)^2$.

The latter geometry has been considered in [4] to geometrically engineer $\mathcal{N} = 1$ four-dimensional supersymmetric gauge theory. After the transition the breaking to $\mathcal{N} = 1$ is achieved by putting N_k units of Ramond flux on the S^3's and the topological string or the matrix model calculates terms in the $\mathcal{N} = 1$ effective gauge theory. In [5] it is already shown that the special geometry relation that determine the tree level (genus zero) topological string amplitude $F^{(0)}$ on the geometry (3.1) arise from the planar diagrams of the corresponding matrix model. The planar loop equation of the matrix model gives the spectral curve of the local geometry and the effective superpotential in a large class of $\mathcal{N} = 1$ gauge theory can be computed exactly by the genus zero amplitude of matrix model or topological string. It is conjectured that higher genus topological B-model string amplitudes should also be computed by higher genus diagrams in the matrix model.

The meaning of the topological string amplitudes $F^{(g>0)}(S_i)$ in the effective theory is as follows. In $\mathcal{N} = 2$ supergravity action they determine the exact moduli dependence of the F-terms

$$\int d^4x \int d^4\theta F^{(g)}(S_i)(\mathcal{W}^{\alpha\beta}\mathcal{W}_{\alpha\beta})^g \tag{3.2}$$

here $\mathcal{W}_{\alpha\beta}$ is the $\mathcal{N} = 2$ graviphoton superfield and the S_i are the $\mathcal{N} = 2$ vector multiplets whose complex scalar field corresponds to the moduli of Calabi-Yau manifold. After

[5] Further generalization of this conjecture to the case of Calabi-Yau geometry with ADE type singularities can also be made [3].

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integrating over the $\mathcal{N}=2$ superspace these terms become the coupling $R_+^2 F_+^{2g-2}$ of the self-dual part of the Ricci tensor R_+ to the selfdual part of the graviphoton field strength F_+.

After breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ by fluxes the topological string amplitudes $F^{(g>0)}(S_i)$ occur in the following two terms of the $\mathcal{N}=1$ action [41 – 43]

$$\Gamma_1 = g\int d^4x \int d^2\theta \mathcal{W}_{\alpha\beta\gamma}\mathcal{W}^{\alpha\beta\gamma}(F_{\delta\xi}F^{\delta\xi})^{g-1}F^{(g)}(S_i), \tag{3.3}$$

$$\Gamma_2 = \int d^4x \int d^2\theta (F_{\alpha\beta}F^{\alpha\beta})^g N_i \frac{\partial F^{(g)}(S_i)}{\partial S_i}\ . \tag{3.4}$$

Here $\mathcal{W}_{\alpha\beta\gamma}$ are now $\mathcal{N}=1$ gravitino multiplet, and S_i are the $\mathcal{N}=1$ glueball chiral superfields coming from the original $\mathcal{N}=2$ vector multiplets [41]. The graviphoton field $F_{\alpha\beta}$ can be treated as background field in the $\mathcal{N}=1$ theory. In [41] a C-deformation is introduced to deform the anti-commutation relation of gluino ψ_α to the followings

$$\{\psi_\alpha, \psi_\beta\} = 2F_{\alpha\beta} \tag{3.5}$$

It is shown that the effect of turning on the graviphoton background $F_{\alpha\beta}$ can be captured by this C-deformation, and the $F^{(g)}$ in the second contribution Γ_2 (3.4) are computed by matrix models at genus g. We can also see that the first term Γ_1 in (3.3) contributes at genus $g=1$ even there is no graviphoton background $F_{\alpha\beta}$. It is shown in [42] that this genus one contribution is also computed by matrix model genus one diagrams. There are also some other types of gravitational corrections besides (3.3), (3.4) of the form $W_{\alpha\beta\gamma}W^{\alpha\beta\gamma}S^n$ from planar diagrams, which become trivial after the extremization of the glue ball superfield S [43]. Our results confirm these very interesting ideas in [41 – 43] by a first direct tests of the connection between topological strings and matrix models at higher genus without using the superspace techniques in the effective gauge theory.

3.2 The two cut geometry and the tree level and genus one amplitudes

We consider now the case of a cubic potential $W(x) = \frac{m}{2}x^2 + \frac{g}{3}x^3$ in the Dijkgraaf-Vafa geometry (3.1). The degree one polynomial $f(x) = \mu_1 x + \mu_0$ splits the double zeros of $W'(x)^2$ at $x = a_1$ and $x = a_2$ to the four roots $a_1^\pm, a_2^\pm$ of the equation

$$W'(x)^2 + f(x) = 0 \tag{3.6}$$

We adopt the notation of [8, 4], and change variable $(a_1^-, a_1^+, a_2^-, a_2^+) \equiv (x_1, x_2, x_3, x_4) \to (z_1, z_2, Q, I)$ where

$$\begin{aligned} z_1 &\equiv \frac{1}{4}(x_2 - x_1)^2, \qquad\qquad z_2 \equiv \frac{1}{4}(x_4 - x_3)^2 \\ Q &\equiv \frac{1}{2}(x_1 + x_2 + x_3 + x_4) = -\frac{m}{g} \\ I &\equiv \frac{1}{4}[(x_3 + x_4) - (x_1 + x_2)]^2 = \left(\frac{m}{g}\right)^2 - 2(z_1 - z_2)\ . \end{aligned} \tag{3.7}$$

The local Calabi-Yau geometry (3.6) depends really only two complex structure deformations. We will use (z_1, z_2) or below the A-periods S_1 and S_2 to parameterize them. The dependence of the genus zero super- and the higher genus potentials $F^{(g)}$ can be reconstructed from scaling laws and dimensional considerations. In particular we frequently set $g, m = 1$.

The fundamental periods of the local geometry are

$$S_i = \frac{1}{2\pi i}\int_{a_i^-}^{a_i^+} \lambda, \qquad \Pi_i = \frac{1}{2\pi i}\int_{a_i^+}^{\Lambda} \lambda\,,$$

where $i = 1, 2$ and $\lambda = dx\sqrt{W'(x)^2 + f(x)}$ is a meromorphic differential which emerges after integrating the holomorphic Calabi-Yau (3,0)-form over an S^2 fibre direction of the $S^{3'}$s in (3.1) [4]. In [4] the integrals where further calculated perturbatively for small z_i. This limit corresponds to vanishing $S^{3'}s$ and is suitable for the perturbative matrix model expansion. Solving the B-model and fixing its ambiguity requires a global understanding of the complex moduli space in (z_1, z_2). We therefore derive the Picard-Fuchs equations and use them to explore the global properties of the integrals S_i and Π_i.

We find that derivatives of λ w.r.t. z_i up to second order multiplied with suitable polynomials in z_i combine to a total derivative, i.e. $\mathcal{L}_i\lambda = \sum_{k+l\leq 2} p^{(k,l)}(\underline{z})\partial_{z_1}^k\partial_{z_2}^l\lambda = \mathrm{d}g_i(x, z_1, z_2)$. Naively the differential ideal with this property is generated by three independent differential operators $\mathcal{L}_i$. However $\mathrm{d}g_i(x, z_1, z_2)$ is a meromorphic differential with non-vanishing residua, hence one cannot conclude from the exactness that $\mathcal{L}_i\int_\Gamma\lambda = 0$. For the following two operators the residua vanish

$$\begin{aligned}\mathcal{L}_1 =\ & 2z_1(2z_1+6z_2-1)\partial_{z_1}^2 + (1-10z_1+12{z_1}^2+4z_1z_2)\partial_{z_1}\partial_{z_2} + (3-2z_1-6z_2)\partial_{z_1}\\ & +(1\leftrightarrow 2) \qquad (3.8)\\ \mathcal{L}_2 =\ & (2z_1+2z_2-1)[-2z_1(1-4(z_1+z_2)+5{z_1}^2-2z_1z_2-3{z_2}^2)\partial_{z_1}^2 + (z_1+z_2)\\ & (1-8z_1+6{z_1}^2-6z_1z_2)\partial_{z_1}\partial_{z_2}] + (z_1(7-18z_2+26{z_2}^2+46{z_1}^2+z_1(62z_2-39))\\ & -3z_2(1-3z_2+2z_2^2)))\partial_{z_1} - 3(1-12z_1+18{z_1}^2+14z_1z_2) + (1\leftrightarrow 2)\ . \qquad (3.9)\end{aligned}$$

These Picard-Fuchs operators annihilate the periods and fix their expansion up to linear combinations. The discriminant of these differential operators has the following components

$$\begin{aligned} C_1 : z_1 = 0, \quad C_2 :\ z_2 = 0, \quad I :\ I = 1-2z_1-2z_2 = 0,\\ J : J &= (x_1-x_2)(x_2-x_3)(x_2-x_4)(x_1-x_4)\\ &= (1-3z_1)^2 - 6z_2 + 9z_2^2 + 14z_1z_2\\ &= 0, \end{aligned} \qquad (3.10)$$

whose schematic intersection after a suitable desingularisation of three order tangencies is depicted in figure 1.

According to the Dijkgraaf-Vafa correspondence the periods S_i are identified with the filling fractions N_i, with $\sum_{i=1}^n N_i = N$, of eigenvalues in the large N limit of the dual matrix model, and it is shown that the special geometry relation and Picard-Fuchs equations are reproduced in the planar limit of the matrix model [5] and the genus zero topological

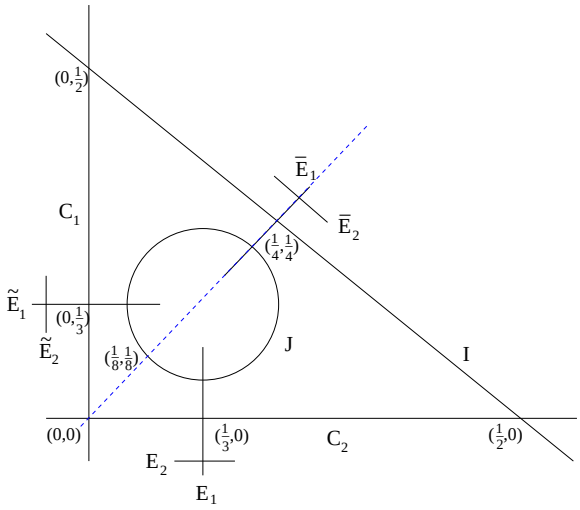

Figure 1: Divisors in the moduli space of B-model, $(z_1, z_2) = (0, 0)$ is the matrix model expansion point and along the dash line one has enhanced $N = 2$ SUSY.

string amplitude $F^{(0)}$ follows from integrating the special geometry relation $\frac{\partial F^{(0)}}{\partial S_i} = \Pi_i$. To analyze the exact effective superpotential $W_{\text{eff}} = 2\pi i \sum_i (N_i \Pi_i + \alpha_i S_i)$, where N_i and α_i are 3-form flux quanta through the A_i and B_i cycles respectively, globally we need to the periods throughout the complex moduli space. This is done in appendix A, where we also find that there is no point where the periods degenerate with leading behavior that is quadratically in the logarithms. The latter degeneration is typical for a large volume or maximal unipotent point in the moduli space of local Calabi-Yau spaces.

Also at one loop the conjecture holds[6] [8, 10]. The B-model expression for the one loop free energy $F^{(1)}$ obtained by integrating (4.3) using (4.1) and fixing integration constants it turns out to be [8]

$$F^{(1)} = \frac{1}{2} \log \left(\det \left(\frac{\partial z_i}{\partial S_j} \right) (z_1 z_2)^{-\frac{1}{6}} I J^{\frac{2}{3}} \right) , \tag{3.11}$$

where we obtain the periods S_i in terms of complex structure moduli z_i and the corresponding inverse series

$$\begin{aligned} S_1(z_1, z_2, g) &= \frac{1}{4} z_1 - \frac{g^2}{8} z_1 (2z_1 + 3z_2) - \frac{g^4}{32} z_1 (4z_1^2 + 13 z_1 z_2 + 9 z_2^2) + \mathcal{O}(g^6) \\ S_2(z_1, z_2, g) &= -S_1(z_2, z_1, g) \\ z_1(S_1, S_2, g) &= 4S_1 + 8g^2 S_1 (2S_1 - 3S_2) + 8g^4 (20 S_1^2 - 67 S_1 S_2 + 39 S_2^2) + \mathcal{O}(g^6) \\ z_2(S_1, S_2, g) &= z_1(-S_2, -S_1, g) \end{aligned} \tag{3.12}$$

from the two power series solutions to (3.9) at $(z_1, z_2) = (0, 0)$. Identifying S_i with the filling fractions N_i we get the genus one contributions to (B.6). Note that the integration constants c_i in $(z_1 z_2)^{c_1} I^{c_2} J^{c_3}$, which fix the behavior of $F^{(1)}$ at the discriminant components

[6] The one loop-test in [10] and the higher loops tests [8] are made at the $N = 2$ point $S_1 = -S_2$.

are global data do not depend on the base point $(z_1, z_2) = (0, 0)$ or the holomorphic limit $\bar{S}_{\bar{\imath}} \to 0$ taken at this base point to obtain the matrix model expansion. The coefficient $c_1 = -\frac{1}{6}$ at the conifold (shrinking S^3) is a universal property of the topological B-model.

To solve the B-model recursion we need the genus zero three point functions, which are rational functions in complex structure moduli.

$$C_{z_1 z_1 z_1} = \frac{1 - (6z_1 + 5z_2)g^2 + 3(3z_1^2 + 3z_1 z_2 + 2z_2^2)g^4}{16 g^4 z_1 I^2}, \quad C_{z_1 z_1 z_2} = \frac{1 - (3z_1 + 5z_2)g^2}{16 g^2 I^2} \ . \tag{3.13}$$

The three point functions $C_{z_i z_j z_k}$ are symmetric in ijk and from symmetry consideration follows $C_{z_2 z_2 z_2} = C_{z_1 z_1 z_1}(z_1 \leftrightarrow z_2)$ as well as $C_{z_2 z_1 z_1} = C_{z_1 z_1 z_1}(z_1 \leftrightarrow z_2)$.

The corresponding matrix model is a Hermitian matrix model with the cubic potential $W(\Phi) = \frac{1}{2}\Phi^2 + \frac{g}{3}\Phi^3$ for a rank N Hermitian matrix Φ. The partition function and free energy F of the model are

$$Z = e^F = \frac{1}{Vol(U(N))} \int D\Phi \ e^{-W(\Phi)} \tag{3.14}$$

In the large N limit the eigenvalues distribute around the two critical points 0 and $-\frac{1}{g}$ of the potential and form two cuts. We consider the metastable vacuum where with N_1 eigenvalues at 0 and N_2 eigenvalues at $-\frac{1}{g}$. This is a two-cut solution of the matrix model with N_1 and N_2 fixed and subject to the condition $N_1 + N_2 = N$. In the large N limit the free energy of the matrix model has genus expansion in $1/N^2$, and at each genus there is a perturbative expansion by the t'Hooft coupling constant gN. Dijkgraaf and Vafa conjecture that the free energy of the matrix model at each genus is matched to the topological string amplitudes on the local Calabi-Yau geometry (3.1), by identifying of the periods S_i in the geometry with the eigenvalue filling fractions N_i. In appendix B we review more details of the matrix model calculations of the free energy.

4. The holomorphic anomaly equations

The key to the solution of the topological B-model are the holomorphic anomaly equations. To solve them recursively one needs in general to derive three types of propagators S^{ij}, $S^{i\phi}$ and $S^{\phi\phi}$ [1]. In local geometries $S^{i\phi}$ and $S^{\phi\phi}$ can be gauged to zero [26] and the derivation of the propagators in the multimoduli case is discussed in [1, 38, 44].

4.1 The holomorphic anomaly recursions

The geometry on the complex structure moduli space z_i of Calabi-Yau is a special Kahler geometry. Its metric, connection and curvature are determined by the Kahler potential K by the well-known formula, $G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$, $\Gamma^i_{lm} = -G^{i\bar{k}} \partial_l G_{\bar{k}m}$ and $R^k_{i\bar{j}l} = -\bar{\partial}_{\bar{j}} \Gamma^k_{il}$. They have a well-known special geometry relation with the three point Yukawa coupling $C_{ijk} = D_i D_j D_k F^{(0)}$, which comes from the tt^* equation [45] and can be thought of as the holomorphic anomaly equation at genus zero

$$R^k_{i\bar{j}l} = G_{i\bar{j}} \delta^k_l + G_{k\bar{j}} \delta^k_i - C_{ilm} \bar{C}^{km}_{\bar{j}} \tag{4.1}$$

At genus one and higher genus the topological string amplitudes has a holomorphic anomaly, and the anti-holomorphic dependence of the genus g free energy is related to lower genus free energy by the holomorphic anomaly equation [1]

$$\partial_i \bar{\partial}_{\bar{j}} F^{(1)} = \frac{1}{2} C_{ikl} \bar{C}^{kl}_{\bar{j}} - \left(\frac{\chi}{24} - 1\right) G_{i\bar{j}} \tag{4.2}$$

$$\bar{\partial}_{\bar{i}} F^{(g)} = \frac{1}{2} \bar{C}^{jk}_{\bar{i}} \left(D_j D_k F^{(g-1)} + \sum_{r=1}^{g-1} D_j F^{(r)} D_k F^{(g-r)} \right) \qquad g \geq 2 \ . \tag{4.3}$$

Here the $\bar{C}^{jk}_{\bar{i}} = e^{2K} \bar{C}_{\bar{i}\bar{j}\bar{k}} G^{\bar{j}j} G^{\bar{k}k}$ define the propagators S^{ij} as $\bar{\partial}_{\bar{i}} S^{jk} = \bar{C}^{jk}_{\bar{i}}$. The holomorphic equation can be integrated and represented as graphic Feynman rules to give the higher genus free energy in terms of lower genus up to a holomorphic ambiguity. The propagators can be solved by integrating its defining relation and use the special geometry relation (4.1). One finds

$$S^{ij} C_{jkl} = \delta^i_l \partial_k K + \delta^i_k \partial_l K + \Gamma^i_{kl} + f^i_{kl} \tag{4.4}$$

here f^i_{kl} are ambiguous integration constants, and they are meromorphic rational functions of the complex structure moduli z_i with poles at discriminant points of the moduli space. Suppose there are n complex structure moduli, then there are $\frac{1}{2}n^2(n+1)$ equations for $\frac{1}{2}n(n+1)$ propagators S^{ij}. In the case of one modulus, the number of equations and propagators are the same, so the meromorphic functions f^i_{jk} can be just set to zero. However, in the multi-moduli case we consider, the equations over determine the propagators, so we have to choose the ambiguity f^i_{jk} properly to satisfy some constrains and ensure we can solve for the propagators.

There are certain simplification in B-model calculations for the case of non-compact local Calabi-Yau manifold. In this case there is a choice of gauge such that the Kahler potential K and metric over the moduli space $G_{S_k \bar{S}_{\bar{j}}}$ in the S_i coordinates is a constant in holomorphic limit, and the dilaton component in the propagators vanish. So in the holomorphic limit $\bar{S}_{\bar{i}} \to 0$ the connection vanishes and covariant derivative in the S_i coordinates is just ordinary derivative. This also makes the topological string amplitudes entirely independent of quantities such as Euler number, Chern classes of the Calabi-Yau, which will need to be regularized in the non-compact case. In this case, the metric and the connection in the z_i coordinates are

$$\begin{aligned} G_{z_i \bar{z}_{\bar{j}}} &= \frac{\partial S_k}{\partial z_i} C_{k\bar{j}} \\ \Gamma^{z_i}_{z_j z_k} &= -\frac{\partial z_i}{\partial S_l} \frac{\partial^2 S_l}{\partial z_j \partial z_k} \end{aligned} \tag{4.5}$$

where $C_{k\bar{j}}$ are constant in the holomorphic limit.

4.2 Propagators and Dijkgraaf-Vafa conjecture at higher genus

The geometry we consider has two complex structure moduli. In order to solve the propagators, the holomorphic ambiguity f^i_{jk} have to satisfy 3 constrain equations by eliminating

propagators S^{ij} in (4.4). These constrains equations are rational functions of the complex structure moduli z_i,[7] and we are able to find a rational solution for the f^i_{jk}

$$f^1_{11} = -\Big[6-(49z_1+48z_2)g^2+(163z_1^2+219z_1z_2+126z_2^2)g^4 \\ +(210z_1^3+304z_1^2z_2+242z_1z_2^2+108z_2^3)g^6)\Big]/(20g^6z_1I^2J) \tag{4.6}$$

$$f^1_{12} = -\Big[29-(79z_1+157z_2)g^2+(10z_1^2+260z_1z_2+210z_2^2)g^4]/(20g^4I^2J) \tag{4.7}$$

$$f^1_{22} = \Big[7-(55z_1+68z_2)g^2+(142z_1^2+315z_1z_2+219z_2^2)g^4 \\ -(120z_1^3+338z_1^2z_2+492z_1z_2^2+234z_2^3)g^6\Big]/(20g^6z_2I^2J).$$

By definition $f^i_{jk} = f_{kj}$ and the $(z_1 \leftrightarrow z_2)$ symmetry determines the other f^i_{jk}, e.g. $f^2_{22} = f^1_{11}(z_1 \leftrightarrow z_2)$ e.t.c. Note that the discriminant factors in (3.10) should be the only singularities appearing in the denominator for the ansatz of holomorphic ambiguities and in fact all appear.

The holomorphic anomaly equation at genus one can be integrated to give the Ray-Singer torsion of the target manifold. The genus one free energy can also be expressed in terms of genus zero three point functions and the propagators as follows

$$\partial_i F^{(1)} = \frac{1}{2}S^{jk}C_{ijk} + \partial_i \sum_r a_r \log(\Delta_r) \tag{4.8}$$

here a_r are constants and Δ_r are various discriminants of the local geometry. The holomorphic ambiguity f^i_{jk} should give a solution of the propagators S^{ij} that satisfies the above consistency check (4.8). We have chosen our ansatz of the holomorphic ambiguity (4.6) that satisfies the (4.8) with the constants $a_1 = 1/15$, $a_2 = 2/15$

$$\partial_i F^{(1)} = \frac{1}{2}S^{jk}C_{ijk} + \partial_i\left(\frac{1}{15}\log(z_1z_2) + \frac{2}{15}\log(\Delta_2)\right) \tag{4.9}$$

This choice of ansatz is convenient in the sense that it leads to the correct leading behavior in genus two, so it is easier for us to fix the genus two holomorphic ambiguity there.

In local geometry the genus two topological free energy can be integrated from the holomorphic anomaly equation. It is

$$F^{(2)} = -\frac{1}{8}S^{ij}S^{kl}F^{(0)}_{ijkl} + \frac{1}{2}S^{ij}F^{(1)}_{ij} - \frac{1}{2}S^{ij}S^{kl}F^{(1)}_{i}F^{(0)}_{jkl} + \frac{1}{2}S^{ij}F^{(1)}_{i}F^{(1)}_{j} \\ +\frac{1}{12}S^{ij}S^{kl}S^{mn}F^{(0)}_{ikm}F^{(0)}_{jln} + \frac{1}{8}S^{ij}S^{kl}S^{mn}F^{(0)}_{ijk}F^{(0)}_{lmn} + f^{(2)} \tag{4.10}$$

We fix the genus two holomorphic ambiguity $f^{(2)}$ with some initial data from matrix model calculations

$$f^{(2)} = -\Big[1253-10503(z_1+z_2)g^2+27(1081z_1^2+950z_1z_2+1081z_2^2)g^4 \\ -26865(z_1+z_2)(z_1-z_2)^2g^6\Big]/(9000g^4z_1z_2J^2), \tag{4.11}$$

[7] We note that while genus zero three point functions are rational functions of the complex structure moduli z_i, the connections Γ^i_{jk} are generally not rational functions. They combine to give rise to rational equations for holomorphic ambiguity f^i_{jk}.

Once the holomorphic ambiguity is fixed, we can compute the genus two free energy to very high order using (4.10). The topological string approach is much more advantageous than direct matrix model calculations where it is hard to compute to free energy at higher orders (see appendix B for more details). After some extensive computer running time, we are able to make many checks of the topological string predictions (4.12) below for the genus two free energy.

$$\begin{aligned}
F^{(2)} = & -\frac{1}{240}\left(\frac{1}{{N_1}^2}+\frac{1}{{N_2}^2}\right)+\left(\frac{35}{6}N_1-\frac{35}{6}N_2\right)g^6 \\
& +(338{N_1}^2-1632N_1N_2+338{N_2}^2)g^8+\left(\frac{66132}{5}{N_1}^3-120880{N_1}^2N_2+\cdots\right)g^{10} \\
& +\left(\frac{1305280}{3}{N_1}^4-\frac{18059582}{3}{N_1}^3N_2+11824166{N_1}^2{N_2}^2-\cdots\right)g^{12} \\
& +\left(12963696{N_1}^5-244438427{N_1}^4N_2+745362156{N_1}^3{N_2}^2-\cdots\right)g^{14} \\
& +\left(362264064{N_1}^6-\frac{35002369227}{4}{N_1}^5N_2+\frac{148144671957}{4}{N_1}^4{N_2}^2\right. \\
& \qquad \left.-57597548553{N_1}^3{N_2}^3+\cdots\right)g^{16} \\
& +\left(\frac{29035470208}{3}{N_1}^7-\frac{862430780350}{3}{N_1}^6N_2+1580964252892{N_1}^5{N_2}^2\right. \\
& \qquad \left.-\frac{10256675032550}{3}{N_1}^4{N_2}^3+\cdots\right)g^{18} \\
& +\left(\frac{1250634104832}{5}{N_1}^8-\frac{44363662176978}{5}{N_1}^7N_2+\frac{303466060570354}{5}{N_1}^6{N_2}^2\right. \\
& \left.-\frac{854152004682126}{5}{N_1}^5{N_2}^3+237637137780236{N_1}^4{N_2}^4-\cdots\right)g^{20} \qquad (4.12)
\end{aligned}$$

The main difficulty in the B-model calculations is to fix the holomorphic ambiguities at each genus. Also the Feynman rules that solve the holomorphic equations quickly become very complicated. Here we push the calculations only to genus two in our calculations, since there are less new conceptual issues beyond that.

5. Conclusion

The B-model iteration in the genus bears some resemblance to the procedure compute higher genus free energy and resolvent of the matrix models for one-cut solution in [12] and generalized to multi-cut solution in [13, 48, 49], where the iteration equation is obtained by doing $1/N$ expansion in the loop equations, and looks similar to the holomorphic anomaly equation in topological strings.

From the hermitian matrix model point of view the anti-holomorphicity is very unnatural. The holomorphic anomaly equations were re-interpreted in [2] as infinitesimal manifestation of the fact that the topological string partition function transforms as a wave function under change of polarization in the middle cohomology of the target space.

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Using this picture the failure of holomorphicity can be traded against a failure of modularity with a similar iteration [50], which makes the connection more naturally.

It would be very interesting to compare this latter iteration with the iterations [48, 49] in detail, since this can in principle fix the holomorphic ambiguity in B-model calculations. Fixing the holomorphic ambiguity systematically is one of the main difficulties for topological string calculations in many other models. We hope further studies will clarify these issues and provide valuable lessons in fixing holomorphic ambiguity in more general models.

Acknowledgments

A.K. likes to thank M. Aganagic, V. Bouchard, M. Mariño, S. Theisen and D. Zagier for discussions, Nakajima for providing expressions for the higher instanton numbers and M. Matone for a correspondence. M.H. would like to thank Simons Workshop on Mathematics and Physics 2005, Hangzhou 2005 Winter Workshop on String Theory, MSRI at Berkeley for hospitality during parts of the work. Our work is supported by DOE-FG02-95ER40896.

A. Moduli space and monodromy of the two cut matrix model

A.1 Compactification of the moduli space and local expansions

The aim of this section is to obtain the periods everywhere in the moduli space and to determine the monodromies. For the compactification of the moduli space we use the projective space $\mathbb{CP}^2$ with homogeneous coordinates $(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ and identify the $\tilde{z}_3 \neq 0$ patch with

$$z_1 = \frac{\tilde{z}_1}{\tilde{z}_3}, \quad z_2 = \frac{\tilde{z}_2}{\tilde{z}_3} \ . \tag{A.1}$$

In addition to the divisors listed in (3.10) we get now a $\mathbb{CP}^1$ divisor at infinity, at which the periods turn out to be non-singular. We calculated the local expansion near all normal crossing divisors and determined the local monodromy. By analytic continuation we determined the global mondromy. One remarkable aspect of the geometry is that there is no point in the moduli space where at least one of the periods degenerates with double logarithm, which would correspond to the normal large complex structure point of a local geometry at which the mirror expansion in the large Kähler coordinates leads to a convergent instanton sum.

Suppose we expand the Picard-Fuchs equation around a common point of two singular divisors $\Delta_1(z_1, z_2) = 0$ and $\Delta_2(z_1, z_2) = 0$. In order to find complete solutions of the Picard-Fuchs equation, one must choose a good local coordinate around these singular points. The technique for choosing good local coordinates is quite standard in algebraic geometry. For our two parameter model, there are two possible cases:

- $\det(\frac{\partial \Delta_i}{\partial z_j}) \neq 0$, then the point $\Delta_1(z_1, z_2) = \Delta_2(z_1, z_2) = 0$ is called the point of normal intersection of divisor $\Delta_1 = 0$ and $\Delta_2 = 0$. In this case a choice of good local coordinates is simply (Δ_1, Δ_2).

- $\det(\frac{\partial \Delta_i}{\partial z_j}) = 0$, then this is called a point of tangency of divisor $\Delta_1 = 0$ and $\Delta_2 = 0$. In this case one will not be able to find all solutions around the point of tangency with the choice of local coordinates (Δ_1, Δ_2). We will encounter a very common situation in which the divisors have the following form

$$\begin{aligned} \Delta_1 &= a^2 + bc \\ \Delta_2 &= b \end{aligned} \tag{A.2}$$

  here a, b and c are degree one polynomial of complex structure moduli z_1 and z_2. The standard technique in algebraic geometry is to introduce two exceptional divisors to resolve the point of tangency. It turns out that a choice of good local coordinate in this case is $(a, \frac{b}{a^2})$. In our analysis we will follow the standard procedure and use this good local coordinates.

We list the asymptotic solutions of Picard-Fuchs equations and their monodromy at various singular points of the divisors (3.10) in the moduli space. Some of the singular points can be obtained by exchanging z_1 and z_2, and we only list once these symmetric singular points.

(i) $C_1 \cap C_2$. his is the matrix model point we have tested Dijkgraaf-Vafa conjecture at higher genus. Th good choice of local coordinates is simply (z_1, z_2). For completeness we list the periods

$$\begin{aligned} f_1 &= \frac{z_1}{4} - \frac{1}{8} z_1(2z_1 + 3z_2) + \cdots \\ f_2 &= -\frac{z_2}{4} + \frac{1}{8} z_2(3z_1 + 2z_2) + \cdots \\ f_3 &= f_1 \log(z_1) + \frac{1}{12} + \frac{z_1}{8} - \frac{1}{16}(-4z_1^2 + z_1 z_2 + 5z_2^2) + \cdots \\ f_4 &= f_2 \log(z_2) - \frac{1}{12} - \frac{z_2}{8} - \frac{1}{16}(5z_1^2 + z_1 z_2 - 4z_2^2) + \cdots \end{aligned} \tag{A.3}$$

(ii) $C_1 \cap I$. The intersection point is at $(z_1, z_2) = (0, \frac{1}{2})$, and the choice of good local coordinates is $(x_1, x_2) = (z_1, 1 - 2z_1 - 2z_2)$. The asymptotic solutions for periods are

$$\begin{aligned} f_1 &= \sqrt{x_2}(1 - 4x_1 - x_2) \\ f_2 &= \sqrt{x_2}\left(x_1 + \frac{x_2}{12} - \left(2x_1^2 - \frac{5}{3} x_1 x_2 - \frac{x_2^2}{8} \right) + \cdots \right) \\ f_3 &= x_1 \left(1 - \frac{5}{4} x_1 - \frac{3}{2} x_2 + \cdots \right) \\ f_4 &= f_3 \log(x_1) + \frac{2}{3} + x_2 + \left(\frac{7}{8} x_1^2 - 3x_1 x_2 - \frac{13}{12} x_2^2 \right) + \cdots \end{aligned} \tag{A.4}$$

(iii) $C_1 \cap J$. This is a point of tangency of the two divisors at $(z_1, z_2) = (0, \frac{1}{3})$. The singular factor J can be written as

$$J = 1 - 6z_1 - 6z_2 + 9z_1^2 + 14 z_1 z_2 + 9 z_2^2 = 9\left(z_2 - \frac{1}{3} \right)^2 + z_1(-6 + 14 z_2 + 9 z_1) \tag{A.5}$$

Following our discussion above we see a good choice coordinates is $(x_1, x_2) = (\frac{z_1}{(z_2-\frac{1}{3})^2}, \frac{1}{3} - z_2)$. This is the local coordinates around the intersection of the blow up divisor with the divisor C_1. The asymptotic expansion for the periods are

$$\begin{aligned} f_1 &= x_1 x_2^{\frac{5}{2}} \left(1 - \frac{1}{216}x_1 - \frac{23}{72}x_1 x_2 - \frac{5}{46656}x_2^2 + \cdots\right) \\ f_2 &= x_2^2 \left(1 + \frac{4}{9}x_1 + 4x_2 + \cdots\right) \\ f_3 &= 1 + 81x_2^3 - 108x_1 x_2^3 - \frac{729}{8}x_2^4 + \cdots \\ f_4 &= f_1 \log(x_1) + x_2^{\frac{5}{2}} \left(\frac{36}{5} - \frac{108}{7}x_2 + \cdots\right) \end{aligned} \tag{A.6}$$

We can also solve the Picard-Fuchs equations with the local coordinates around the intersection of the blow up divisor with the singular divisor J. This will be useful later on when we try to match the basis and derive the monodromy of the divisor J. The good choice of local coordinates around this point is

$$\begin{aligned} x_1 &= \frac{z_1(6 - 14z_2 - 9z_1)}{9(z_2 - \frac{1}{3})^2} - 1, \\ x_2 &= \frac{1}{3} - z_2 \end{aligned} \tag{A.7}$$

We find the asymptotic expansion for the periods with this coordinates

$$\begin{aligned} f_1 &= x_1^2 x_2^{\frac{5}{2}} \left(1 - \frac{5}{16}x_1 + \frac{33}{8}x_2 + \cdots\right) \\ f_2 &= x_2^2 \left(1 + \frac{3}{4}x_1 - \frac{43}{2}x_2 + \cdots\right) \\ f_3 &= 1 - 648x_2^3 + \cdots \\ f_4 &= f_1 \log(x_1) + x_2^{\frac{5}{2}} \left(\frac{512}{15} + 32x_1 - \frac{3008}{7}x_2 + \cdots\right) \end{aligned} \tag{A.8}$$

(iv) $I \cap J$. This is a point of tangency between the divisors at $(z_1, z_2) = (\frac{1}{4}, \frac{1}{4})$. We write the singular factor J as

$$J = (z_1 - z_2)^2 + (1 - 2z_1 - 2z_2)(1 - 4z_1 - 4z_2) \tag{A.9}$$

A good choice of local coordinates is $(x_1, x_2) = ((z_1 - z_2)^2, \frac{1-2z_2-2z_2}{(z_1-z_2)^2})$. The asymptotic solutions for the periods are

$$\begin{aligned} f_1 &= x_1 \left(1 + \frac{3}{8}x_1 + x_2 + \frac{35}{64}x_1^2 + \frac{1}{4}x_1 x_2 + \cdots\right) \\ f_2 &= x_1 \sqrt{x_2} \\ f_3 &= f_1 \log(x_1) + \frac{4}{3} + \frac{23}{16}x_1^2 + 3x_1 x_2 + \cdots \\ f_4 &= f_2 \log(x_1) + x_1 x_2^{\frac{3}{2}} \left(\frac{1}{3} + \frac{1}{30}x_2 + \cdots\right) \end{aligned} \tag{A.10}$$

(v) $C_1 \cap C_\infty$. In the homogeneous coordinate $(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ the divisor C_1 is $\tilde{z}_1 = 0$, so the good local coordinates is $(x_1, x_2) = (\tilde{z}_1, \tilde{z}_3)$. Since at the intersection $\tilde{z}_2 \neq 0$, we can choose $\tilde{z}_2 = 1$ and use the relation (A.1) to find $(x_1, x_2) = (\frac{z_1}{z_2}, \frac{1}{z_2})$. The asymptotic solutions for the periods are

$$
\begin{aligned}
f_1 &= x_1 x_2^{-\frac{3}{2}} \left(1 - \frac{23}{72} x_1 - \frac{1}{6} x_2 + \cdots \right) \\
f_2 &= x_2^{-\frac{3}{2}} \left(1 - \frac{1}{4} x_2 - \frac{67}{144} x_1^2 + \frac{7}{24} x_1 x_2 - \frac{1}{32} x_2^2 + \cdots \right) \\
f_3 &= f_1 \log(x_1) + x_2^{-\frac{3}{2}} \left(x_2 - \frac{65}{144} x_1^2 + \frac{1}{2} x_1 x_2 - \frac{1}{8} x_2^2 + \cdots \right) \\
f_4 &= (f_1 - 2 f_2) \log(x_2) + x_2^{-\frac{3}{2}} \left(\frac{2}{3} x_2 - \frac{211}{108} x_1^2 + \frac{25}{18} x_1 x_2 - \frac{1}{6} x_2^2 + \cdots \right)
\end{aligned}
$$

(vi) $I \cap C_\infty$. In the homogeneous coordinates $\tilde{z}_1$, $\tilde{z}_2$ and $\tilde{z}_3$, the divisor I is $\tilde{z}_3 - 2\tilde{z}_1 - 2\tilde{z}_2 = 0$. At the intersection with $I \cap C_\infty$ the coordinate $\tilde{z}_2 \neq 0$, we can choose $\tilde{z}_2 = 1$ and use the relation (A.1) to find the good local coordinates

$$
\begin{aligned}
(x_1, x_2) &= (\tilde{z}_3 - 2\tilde{z}_1 - 2\tilde{z}_2, \tilde{z}_3) \\
&= \left(\frac{1 - 2z_1 - 2z_2}{z_2}, \frac{1}{z_2}\right)
\end{aligned}
\tag{A.11}
$$

The asymptotic solutions for the periods are

$$
\begin{aligned}
f_1 &= \sqrt{x_1} x_2^{-\frac{3}{2}} \left(1 + \frac{x_1 - x_2}{4}\right) \\
f_2 &= x_2^{-\frac{3}{2}} \left(1 + \frac{3}{4} x_1 - \frac{5}{64} x_1^2 + \frac{3}{32} x_1 x_2 - \frac{3}{64} x_2^2 + \cdots \right) \\
f_3 &= x_2^{-\frac{3}{2}} (x_1 + x_2) \left(1 + \frac{x_1 - x_2}{8} + \cdots \right) \\
f_4 &= f_1 \log(x_2) + \sqrt{x_1} x_2^{-\frac{3}{2}} \left(- \frac{1}{4} x_1 + \frac{1}{4} x_2 + \frac{13}{480} x_1^2 + \frac{1}{48} x_1 x_2 - \frac{1}{32} x_2^2 + \cdots \right)
\end{aligned}
$$

(vii) $J \cap C_\infty$. In the homogeneous coordinates $\tilde{z}_1$, $\tilde{z}_2$ and $\tilde{z}_3$, the divisor J is $\tilde{z}_3^2 - 6\tilde{z}_1\tilde{z}_3 - 6\tilde{z}_2\tilde{z}_3 + 9\tilde{z}_1^2 + 14\tilde{z}_1\tilde{z}_2 + 9\tilde{z}_2^2 = 0$. At the intersection with $J \cap C_\infty$ the coordinate $\tilde{z}_2 \neq 0$, we can choose $\tilde{z}_2 = 1$ and use the relation (A.1) to find the good local coordinates

$$
\begin{aligned}
(x_1, x_2) &= (\tilde{z}_3^2 - 6\tilde{z}_1\tilde{z}_3 - 6\tilde{z}_2\tilde{z}_3 + 9\tilde{z}_1^2 + 14\tilde{z}_1\tilde{z}_2 + 9\tilde{z}_2^2, \tilde{z}_3) \\
&= \left(\frac{1 - 6z_1 - 6z_2 + 9z_1^2 + 14 z_1 z_2 + 9 z_2^2}{z_2^2}, \frac{1}{z_2}\right)
\end{aligned}
\tag{A.12}
$$

The asymptotic solutions for the periods are

$$
\begin{aligned}
f_1 &= x_1^2 x_2^{-\frac{3}{2}}\left(1+\frac{103-\sqrt{2}i}{576}x_1+\frac{8+\sqrt{2}i}{16}x_2+\cdots\right)\\
f_2 &= x_2^{-\frac{3}{2}}\left(1+\frac{-5-\sqrt{2}i}{64}x_1+\frac{-5-\sqrt{2}i}{16}x_2+\cdots\right)\\
f_3 &= \left(f_1-\frac{8192(43+13\sqrt{2}i)}{2187}f_2\right)\log(x_2)\\
&\quad +x_2^{-\frac{3}{2}}\left(-\frac{256(7+4\sqrt{2}i)}{243}x_1-\frac{1024(2+5\sqrt{2}i)}{81}x_2+\cdots\right)\\
f_4 &= f_1\log x_1+x_2^{-\frac{3}{2}}\left(\frac{512(23-10\sqrt{2}i)}{81}x_2+\cdots\right)
\end{aligned}
\tag{A.13}
$$

A.2 Analytic continuation

The periods of a Calabi-Yau manifold are integrals of the holomorphic three-form over the three-cycles. In the case of the Dijkgraaf-Vafa model, the integrals of the holomorphic three-form over the symplectic three-cycles reduce to integrals of a differential one-form over its branch cuts on the complex plane. For convenience we consider the cubic potential $W(x)=\frac{1}{2}x^2+\frac{1}{3}x^3$ with the cubic coupling g set to one. The A-cycle periods and B-cycle periods are [8]

$$
\begin{aligned}
S_1 &= \frac{1}{2\pi i}\int_{x_3}^{x_4} dx\sqrt{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}\\
S_2 &= -\frac{1}{2\pi i}\int_{x_1}^{x_2} dx\sqrt{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}\\
\Pi_1 &= \frac{1}{2\pi i}\int_{x_3}^{\Lambda_0} dx\sqrt{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}\\
\Pi_2 &= \frac{1}{2\pi i}\int_{x_2}^{\Lambda_0} dx\sqrt{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}
\end{aligned}
\tag{A.14}
$$

The asymptotic expansion of the periods around the origin $(z_1,z_2)=(0,0)$ was considered in [4]. Around this point the roots satisfy $x_1<x_2<x_3<x_4$ and the cuts between x_1,x_2 and between x_3,x_4 shrink to zero sizes. It was found there that the asymptotic expansions of the periods are

$$
\begin{aligned}
S_1 &= \frac{z_1}{4}-\frac{1}{8}z_1(2z_1+3z_2)+\cdots\\
S_2 &= -\frac{z_2}{4}+\frac{1}{8}z_2(3z_1+2z_2)+\cdots\\
2\pi i\Pi_1 &= \frac{\Lambda_0^3}{3}+\frac{\Lambda_0^2}{2}+S_1\log\left(\frac{z_1}{4}\right)-\frac{z_1}{4}-(S_1+S_2)\log(\Lambda_0^2)+\cdots\\
2\pi i\Pi_2 &= \frac{\Lambda_0^3}{3}+\frac{\Lambda_0^2}{2}-\frac{1}{6}+S_2\log\left(\frac{z_2}{4}\right)+\frac{z_2}{4}-(S_1+S_2)\log(\Lambda_0^2)+\cdots
\end{aligned}
\tag{A.15}
$$

[8] The periods are determined only up to a sign ambiguity due to the square root factors in the formula. Here we have taken the proper signs to match the convention of [4].

The Picard-Fuchs equations we derived can determine the constant term in the B-cycle periods Π_i and the cut-off parameter is fixed to be $\Lambda_0 = -\frac{1}{2}$. We can find the monodromy matrices around this point $z_i \to z_i e^{2\pi i}$

$$M_{z_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{z_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{A.16}$$

Since the integrals are done over symplectic cycles, the monodromy matrices are elements of the symplectic group $\mathrm{Sp}(4, \mathcal{Z})$ and satisfy the van Kampen relation $M_{z_1} M_{z_2} = M_{z_2} M_{z_1}$.

Now we want to analytically continue the periods to other points in the complex structure moduli space. The analytic continuation will fix the symplectic basis of periods, which is not available by solving the Picard-Fuchs equation around these points. In order to do the analytic continuation, we must do the integrals in (A.14) exactly. The A-cycle periods S_i and the difference between the two B-cycle periods $\Pi_1 - \Pi_2$ can be written in terms of complete elliptic integrals of the first, second and third kinds, and one of the B-cycle periods involves incomplete elliptic integrals.

We consider analytically continue the periods (A.14) to a singular point $(z_1, z_2) = (0, \frac{1}{3})$ in the moduli space. This is the closest singular point to $(z_1, z_2) = (0, 0)$ in the moduli space. We will use the local coordinate $(\tilde{z}_1, \tilde{z}_2)$ around the intersection of the blow up divisor $F_1 : \frac{1}{3} - z_2 = 0$ and divisor $C_1 : z_1 = 0$ as we did for solving the Picard-Fuchs equation

$$\tilde{z}_1 = \frac{z_1}{(z_2 - \frac{1}{3})^2} \quad \tilde{z}_2 = \frac{1}{3} - z_2 \tag{A.17}$$

We directly compute the asymptotic expansion of one B-cycle period Π_2 and use the asymptotic expansion formulae of the complete elliptic integrals in [46] to obtain the asymptotic formulae for other periods.

For convenience we define a function in terms of complete elliptic integrals of the first kind $K(k^2)$, the second kind $E(k^2)$ and the third kind $\Pi(a^2, k^2)$ as the following

$$\begin{aligned} P(a^2, k^2) = & \frac{1}{48a^4(1-a^2)^2(a^2-k^2)^2} \times \{a^2[3a^8 + 3k^4 - 4a^6(1+k^2) - 4a^2k^2(1+k^2) \\ & +2a^4(2+k^2+2k^4)]E(k^2) + [-3a^{10} - 3k^6 + a^8(4+5k^2) \\ & +a^4k^2(4-10k^2) - 2a^6(2+k^4) + 3a^2k^4(1+2k^2)]K(k^2) \\ & +3[-a^{12} - 5a^8k^2 + 5a^4k^4 + k^6 + 2a^{10}(1+k^2) \\ & -2a^2k^4(1+k^2)]\Pi(a^2, k^2)\} \end{aligned} \tag{A.18}$$

We start from the original matrix model point $(z_1, z_2) = (0, 0)$ in the complex structure moduli space where $x_1 < x_2 < x_3 < x_4$. The expressions for the A-cycle periods can be

found using formulae in [46]. After some algebra we found

$$\begin{aligned}
S_1 &= \frac{1}{2\pi}\int_{x_3}^{x_4} dx\sqrt{(x_4-x)(x-x_3)(x-x_2)(x-x_1)} \\
&= \frac{1}{2\pi}\frac{2(x_4-x_3)(x_2-x_4)(x_4-x_1)^2}{\sqrt{(x_4-x_2)(x_3-x_1)}}P\left(\frac{x_3-x_4}{x_3-x_1},\frac{(x_4-x_3)(x_2-x_1)}{(x_4-x_2)(x_3-x_1)}\right) \\
S_2 &= \frac{1}{2\pi}\int_{x_1}^{x_2} dx\sqrt{(x_4-x)(x_3-x)(x_2-x)(x-x_1)} \\
&= \frac{1}{2\pi}\frac{2(x_4-x_2)(x_2-x_1)(x_3-x_2)^2}{\sqrt{(x_4-x_2)(x_3-x_1)}}P\left(\frac{x_2-x_1}{x_3-x_1},\frac{(x_4-x_3)(x_2-x_1)}{(x_4-x_2)(x_3-x_1)}\right)
\end{aligned} \tag{A.19}$$

and the difference between the two B-cycle periods is

$$\begin{aligned}
2\pi i(\Pi_2-\Pi_1) &= \int_{x_2}^{x_3} dx\sqrt{(x_4-x)(x_3-x)(x-x_2)(x-x_1)} \\
&= \frac{2(x_4-x_2)(x_3-x_2)(x_2-x_1)^2}{\sqrt{(x_4-x_2)(x_3-x_1)}}P\left(\frac{x_3-x_2}{x_3-x_1},\frac{(x_3-x_2)(x_4-x_1)}{(x_4-x_2)(x_3-x_1)}\right)
\end{aligned} \tag{A.20}$$

We will take these exact formulae at the matrix model point and analytically continue to the local coordinate (A.17).

We can also directly compute the asymptotic expansion one of B-cycle periods Π_2 around $(\tilde{z}_1,\tilde{z}_2)=(0,0)$ as follows

$$\begin{aligned}
2\pi i\Pi_2 &= \int_{\sqrt{\frac{1}{3}-\tilde{z}_2}-I}^{\Lambda_0+\frac{1}{2}-\frac{I}{2}} dx\sqrt{\left((x+I)^2-\frac{1}{3}+\tilde{z}_2\right)(x^2-\tilde{z}_1\tilde{z}_2^2)} \\
&= \int_{\sqrt{\frac{1}{3}-\tilde{z}_2}-I}^{\Lambda_0+\frac{1}{2}-\frac{I}{2}} dx\sqrt{\left((x+I)^2-\frac{1}{3}+\tilde{z}_2\right)}\left(x-\frac{\tilde{z}_1\tilde{z}_2^2}{2x}-\frac{(\tilde{z}_1\tilde{z}_2^2)^2}{8x^3}+O(\tilde{z}^4)\right)
\end{aligned} \tag{A.21}$$

where $I=\sqrt{\frac{1}{3}-2\tilde{z}_1\tilde{z}_2^2+2\tilde{z}_2}$. We can compute the integrals exactly for the first few leading terms written above, and expand around the cut off $\Lambda_0=\infty$ keeping only positive powers of Λ_0.

We can now use the expressions for the periods (A.19), (A.20), (A.21) and obtain the asymptotic expansions to a first few orders

$$\begin{aligned}
S_1 &= \frac{\sqrt{3}}{4}f_1 \\
S_2 &= -\frac{\sqrt{3}}{4}f_1+\frac{3\sqrt{3}}{8}f_2-\frac{1}{12\sqrt{3}}f_3 \\
\Pi_1 &= \Pi_2+\frac{1}{2\pi i}\frac{\sqrt{3}}{4}(f_4+(3-\log(2^4 3^3))f_1) \\
2\pi i\Pi_2 &= \frac{\Lambda_0^3}{3}+\frac{\Lambda_0^2}{2}-\frac{1}{12}+\frac{\sqrt{3}}{108}f_3-(S_1+S_2)\log(12\Lambda_0^2)
\end{aligned} \tag{A.22}$$

where f_1, f_2, f_3, f_4 are the asymptotic expansion of the solutions for Picard-Fuchs equation we found earlier

$$\begin{aligned}
f_1 &= \tilde{z}_1\tilde{z}_2^{\frac{5}{2}}\left(1-\frac{1}{216}\tilde{z}_1-\frac{23}{72}\tilde{z}_1\tilde{z}_2-\frac{5}{46656}\tilde{z}_2^2+\cdots\right)\\
f_2 &= \tilde{z}_2^2\left(1+\frac{4}{9}\tilde{z}_1+4\tilde{z}_2+\cdots\right)\\
f_3 &= 1+81\tilde{z}_2^3-108\tilde{z}_1\tilde{z}_2^3-\frac{729}{8}\tilde{z}_2^4+\cdots\\
f_4 &= f_1\log(\tilde{z}_1)+\tilde{z}_2^{\frac{5}{2}}\left(\frac{36}{5}-\frac{108}{7}\tilde{z}_2+\cdots\right)
\end{aligned} \tag{A.23}$$

These asymptotic expressions of periods are linear combinations of the 4 solutions to the Picard-Fuchs equations we found earlier, provided we choose the cut-off constant to be $\Lambda_0 = -\frac{1}{2}$. Thus we have found the canonical basis for the symplectic cycles. It is easy to write down the monodromy matrices around this point

$$M_{\tilde{z}_1} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 1&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad M_{F_1} = \begin{pmatrix} -1&0&0&0\\ 2&1&0&0\\ 0&0&-1&2\\ 0&0&0&1 \end{pmatrix} \tag{A.24}$$

The monodromy around the singular divisor C_1 is the same as before $M_{\tilde{z}_1} = M_{z_1}$. We can also see that the monodromy matrices are elements of $\mathrm{Sp}(4,\mathcal{Z})$ group and satisfy the van Kampen relation $M_{\tilde{z}_1}M_{F_1} = M_{F_1}M_{\tilde{z}_1}$.

In general it is not easy to do the analytic continuation of periods. We use a numerical method to match the basis of solutions of Picard-Fuchs equation at different points of the moduli space. We consider the intersection of the singular divisor J and the blow up divisor $F_1 : \frac{1}{3} - z_2 = 0$. The local coordinate and the solutions for the Picard-Fuchs equation are

$$\begin{aligned}
\tilde{z}_1 &= \frac{z_1(6-14z_2-9z_1)}{9(z_2-\frac{1}{3})^2}-1,\\
\tilde{z}_2 &= \frac{1}{3}-z_2
\end{aligned} \tag{A.25}$$

$$\begin{aligned}
f_1 &= \tilde{z}_1^2\tilde{z}_2^{\frac{5}{2}}\left(1-\frac{5}{16}\tilde{z}_1+\frac{33}{8}\tilde{z}_2+\cdots\right)\\
f_2 &= \tilde{z}_2^2\left(1+\frac{3}{4}\tilde{z}_1-\frac{43}{2}\tilde{z}_2+\cdots\right)\\
f_3 &= 1-648\tilde{z}_2^3+\cdots\\
f_4 &= f_1\log(\tilde{z}_1)+\tilde{z}_2^{\frac{5}{2}}\left(\frac{512}{15}+32\tilde{z}_1-\frac{3008}{7}\tilde{z}_2+\cdots\right)
\end{aligned} \tag{A.26}$$

Using numerical method we find the canonical basis of the periods as the following

$$\begin{aligned}
S_1 &= (-0.14-0.26i)f_1+0.082f_4\\
S_2 &= (0.14+0.26i)f_1+2.6f_2-0.048f_3-0.082f_4\\
\Pi_1 &= -0.26if_1+1.03if_2-0.016f_3\\
\Pi_2 &= 1.03if_2-0.016if_3
\end{aligned} \tag{A.27}$$

The monodromy matrix of the divisor F_1 is the same as we have derived in (A.24). We can now write down the monodromy matrix of the singular divisor J by looking at the transformation around $\tilde{z}_1 \to \tilde{z}_1 e^{2\pi i}$

$$M_J = \begin{pmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.28}$$

For the singular divisor $I : 1 - 2z_1 - 2z_2 = 0$, we find essential singularities instead of simple singularities. This can be seen from the asymptotic behavior of the solutions of the Picard-Fuchs equation at any point in the divisor. We find that the radius of convergence for the asymptotic expansion is zero, i.e. the series is always divergent. This is an interesting new feature of the Dijkgraaf-Vafa model.

B. Matrix model calculations

In this appendix we give some details of the matrix model calculations following the approach in [8, 7]. The cubic matrix model can be expressed in the eigenvalues of the matrix

$$W(\Phi) = \mathrm{tr}\left(\frac{\Phi^2}{2} + \frac{g\Phi^3}{3}\right) = \sum_{i=1}^{N}\left(\frac{\lambda_i^2}{2} + \frac{g\lambda_i^3}{3}\right) \tag{B.1}$$

Then the partition functions Z and free energy F are

$$Z = e^F = \frac{1}{Vol(U(N))}\int D\Phi\ e^{-W(\Phi)} = \frac{1}{N!(2\pi)^N}\int \prod_i d\lambda_i \Delta^2(\lambda) e^{-\sum_{i=1}^{N}(\frac{\lambda_i^2}{2}+\frac{g\lambda_i^3}{3})} \tag{B.2}$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the standard Vandermonde determinant from the measure of the matrix. We expand N_1 eigenvalues around the critical points $a_1 = 0$ and $N_2 = N - N_1$ eigenvalues around the critical points $a_2 = -\frac{1}{g}$. Suppose the fluctuation is μ_i, ν_i

$$\begin{aligned} \lambda_i &= \mu_i, \qquad && i = 1, 2, \cdots N_1 \\ \lambda_{i+N_1} &= -\frac{1}{g} + \nu_i \qquad && i = 1, 2, \cdots N_2 \end{aligned} \tag{B.3}$$

Then the potential and the Vandermonde determinant become

$$W(\Phi) = \sum_{i=1}^{N_1}\left(\frac{\mu_i}{2} + \frac{g\mu^3}{3}\right) - \sum_{i=1}^{N_2}\left(\frac{\nu_i^2}{2} - \frac{g\nu_i^3}{3}\right) + N_2 W\Big(-\frac{1}{g}\Big) \tag{B.4}$$

$$\Delta^2(\lambda) = \prod_{1\le i_1<i_2\le N_1}(\mu_{i_1} - \mu_{i_2})^2 \prod_{1\le j_1<j_2\le N_2}(\nu_{j_1} - \nu_{j_2})^2 \prod_{1\le i\le N_1}\prod_{1\le j\le N_2}\left(\mu_i - \nu_j + \frac{1}{g}\right)^2 \tag{B.5}$$

Now we can treat the expansion around this vacuum as a model with two matrices Φ_1 with eigenvalues μ_i and Φ_2 with eigenvalues ν_i. The interaction terms $\prod_{1\le i\le N_1}\prod_{1\le j\le N_2}(\mu_i - \nu_j + \frac{1}{g})^2$ in the Vandermonde determinant can be exponentiated and written as potential

for the two matrices, then the partition functions can be straightforwardly evaluated by expanding the potential and computing the expectations values of Gaussian matrix model [7]. We note the fluctuation around unstable critical point $-\frac{1}{g}$ has a wrong sign kinetic term $-\frac{\nu_i^2}{2}$. However this model is perturbatively well defined if we treat Φ_1 as a Hermitian matrix and analytically continue Φ_2 to be a anti-Hermitian matrix. Alternatively, one can also determine the perturbative part of the free energy by directly evaluating the Gaussian integral for various values of N_1 and N_2 and solving for the coefficients in the perturbation series. Using this method we are able to push the computations of the free energy to the eighth order, and provide many checks of the topological string calculations in (4.12). The perturbative part of the free energy is the followings

$$
\begin{aligned}
F_{pert} = & -N_2 W(a_2) - 2N_1 N_2 \log(g) \\
& + \left[\left(\frac{2}{3} {N_1}^3 - 5 {N_1}^2 N_2 + 5 N_1 {N_2}^2 - \frac{2}{3} {N_2}^3 \right) + \left(\frac{N_1}{6} - \frac{N_2}{6} \right) \right] g^2 \\
& + \left[\left(\frac{8}{3} {N_1}^4 - \frac{91}{3} {N_1}^3 N_2 + 59 {N_1}^2 {N_2}^2 - \frac{91}{3} N_1 {N_2}^3 + \frac{8}{3} {N_2}^4 \right) \right. \\
& \qquad \left. + \left(\frac{7}{3} {N_1}^2 - \frac{31}{3} N_1 N_2 + \frac{7}{3} {N_2}^2 \right) \right] g^4 \\
& + \left[\left(\frac{56}{3} {N_1}^5 - \frac{871}{3} {N_1}^4 N_2 + \frac{2636}{3} {N_1}^3 {N_2}^2 - \frac{2636}{3} {N_1}^2 {N_2}^3 + \frac{871}{3} N_1 {N_2}^4 - \frac{56}{3} {N_2}^5 \right) \right. \\
& \left. + \left(\frac{332}{9} {N_1}^3 - \frac{923}{3} {N_1}^2 N_2 + \frac{923}{3} N_1 {N_2}^2 - \frac{332}{9} {N_2}^3 \right) + \left(\frac{35}{6} N_1 - \frac{35}{6} N_2 \right) \right] g^6 \\
& + \left[\left(\frac{512}{3} {N_1}^6 - \frac{6823}{2} {N_1}^5 N_2 + \frac{28765}{2} {N_1}^4 {N_2}^2 - \frac{67310}{3} {N_1}^3 {N_2}^3 \pm \cdots \right) \right. \\
& + \left(\frac{1864}{3} {N_1}^4 - \frac{47083}{6} {N_1}^3 N_2 + 15349 {N_1}^2 {N_2}^2 \mp \cdots \right) \\
& \qquad \left. + \left(338 {N_1}^2 - 1632 N_1 N_2 + 338 {N_2}^2 \right) \right] g^8 \\
& + \left[\left(\frac{9152}{5} {N_1}^7 - 45118 {N_1}^6 N_2 + 247980 {N_1}^5 {N_2}^2 - 540378 {N_1}^4 {N_2}^3 \pm \cdots \right) \right. \\
& + \left(\frac{54416}{5} {N_1}^5 - 187528 {N_1}^4 N_2 + 570066 {N_1}^3 {N_2}^2 \mp \cdots \right) \\
& \left. + \left(\frac{66132}{5} {N_1}^3 - 120880 {N_1}^2 N_2 \pm \cdots \right) + \left(\frac{5005}{3} N_1 - \frac{5005}{3} N_2 \right) \right] g^{10} \\
& + \left[\left(\frac{65536}{3} {N_1}^8 - \frac{1933906}{3} {N_1}^7 N_2 + \frac{13258178}{3} {N_1}^6 {N_2}^2 \right. \right. \\
& \qquad \left. - \frac{37761034}{3} {N_1}^5 {N_2}^3 + \frac{52780010}{3} {N_1}^4 {N_2}^4 \mp \cdots \right) \\
& \left(\frac{1762048}{9} {N_1}^6 - \frac{12980560}{3} {N_1}^5 N_2 + \frac{54863776}{3} {N_1}^4 {N_2}^2 - \frac{256344964}{9} {N_1}^3 {N_2}^3 \pm \cdots \right) \\
& \left. \left(\frac{1305280}{3} {N_1}^4 - \frac{18059582}{3} {N_1}^3 N_2 + 11824166 {N_1}^2 {N_2}^2 \mp \cdots \right) \right.
\end{aligned}
$$

$$\left(\frac{1680704}{9}{N_1}^2-\frac{8748896}{9}N_1N_2+\frac{1680704}{9}{N_2}^2\right)\Bigg]g^{12}$$
$$+\Bigg[\Bigg(\frac{5912192}{21}{N_1}^9-\frac{68087967}{7}{N_1}^8N_2+\frac{564130824}{7}{N_1}^7{N_2}^2$$
$$-286953520{N_1}^6{N_2}^3+524636640{N_1}^5{N_2}^4\mp\cdots\Bigg)$$
$$\left(\frac{25136768}{7}{N_1}^7-97692942{N_1}^6N_2+537372540{N_1}^5{N_2}^2-1166263112{N_1}^4{N_2}^3\pm\cdots\right)$$
$$\left(12963696{N_1}^5-244438427{N_1}^4N_2+745362156{N_1}^3{N_2}^2\mp\cdots\right)$$
$$\left(\frac{86388296}{7}{N_1}^3-\frac{855302550}{7}{N_1}^2N_2\pm\cdots\right)+\left(\frac{8083075}{6}N_1-\frac{8083075}{6}N_2\right)\Bigg]g^{14}$$
$$+\Bigg[\Bigg(\frac{11534336}{3}{N_1}^{10}-\frac{1834216417}{12}{N_1}^9N_2+\frac{5978643549}{4}{N_1}^8{N_2}^2-6444922816{N_1}^7{N_2}^3$$
$$+14743157646{N_1}^6{N_2}^4-19289163957{N_1}^5{N_2}^5\pm\cdots\Bigg)$$
$$\Bigg(66841600{N_1}^8-\frac{4347551555}{2}{N_1}^7N_2+\frac{29785674795}{2}{N_1}^6{N_2}^2$$
$$-42151343305{N_1}^5{N_2}^3+58765399140{N_1}^4{N_2}^4\mp\cdots\Bigg)$$
$$\Bigg(362264064{N_1}^6-\frac{35002369227}{4}{N_1}^5N_2+\frac{148144671957}{4}{N_1}^4{N_2}^2$$
$$-57597548553{N_1}^3{N_2}^3\pm\cdots\Bigg)$$
$$\left(\frac{1882324352}{3}{N_1}^4-\frac{28221164683}{3}{N_1}^3N_2+18539047948{N_1}^2{N_2}^2\mp\cdots\right)$$
$$\left(\frac{693764720}{3}{N_1}^2-\frac{3860943680}{3}N_1N_2+\frac{693764720}{3}{N_2}^2\right)\Bigg]g^{16} \tag{B.6}$$

This model also contains a non-perturbative part of free energy defined as the volume factor of the $\mathrm{U}(N)$ gauge group as in [47], where it was computed with the following result

$$F_{n.p.}=\frac{N_1^2}{2}\log(N_1)+\frac{N_2^2}{2}\log(N_2)-\frac{3}{4}(N_1^2+N_2^2)-\frac{1}{12}\log(N_1N_2)$$
$$+2\zeta'(-1)+\sum_{g=2}^{\infty}\frac{B_{2g}}{4g(g-1)}\left(\frac{1}{N_1^{2g-2}}+\frac{1}{N_2^{2g-2}}\right) \tag{B.7}$$

This non-perturbative part of the matrix model has the correct universal leading behavior of Calabi-Yau near the conifold point of its moduli space, as first pointed out in [6] in the context of $c=1$ string compactified at self-dual radius.

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